RIEMANN SURFACES

FREDERIK WITT
UNIVERSITÄT STUTTGART

Ohne Gewähr auf Vollständig- oder Richtigkeit

Abstract. In the same way we can view $\mathbb{R}^2$ as complex space $\mathbb{C}$, a two dimensional *real* surface can be seen as a one dimensional *complex* curve. If we replace smooth functions by holomorphic ones, the surface becomes a genuine geometric object of *complex geometry*. In this way it brings together anaylsis and algebra, and topology and geometry. Moreover, complex geometry is not only confined to surfaces but makes also sense in higher dimensions. It has many important applications to Riemannian and (complex) algebraic geometry as well as to theoretical physics.

Since this is a classical subject there exists a vast and excellent literature. The text here builds essentially on

(i) O. Forster, *Lectures on Riemann surfaces*, Springer;
(ii) W. Fulton, *Algebraic topology*, Springer;
(iii) R. Gunning, *Lectures on Riemann surfaces*, Princeton University Press.

No claim of any originality in the presentation of this material is made.

Contents

1. The category of Riemann surfaces

In order to do geometry we need two things: A topological space and a prefered class of functions. We define Riemann surfaces and holomorphic maps in analogy with smooth manifolds and smooth maps which starts from the notion of a smooth function. Here, we replace smooth functions by *holomorphic* ones, see Appendix A for a brief recap. In this first chapter we construct several examples of Riemann surfaces and holomorphic maps and also discuss various classifications results.

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Unless mentioned otherwise,

X will be a **surface**, that is, a connected second countable Hausdorff space such that every point $a \in X$ admits an open neighbourhood homeomomorphic to $\mathbb{R}^2$.

We note that such a topological space is always metrisable, that is, it is the topology of open balls for some distance function (or metric) d. We denote by $D_r(z)$ an open ball of radius r around $z \in X$. If $X = \mathbb{C}$ and $z = 0$ we simply write D_r for $D_r(0)$. For more specific results on the topology of surfaces we require, see Appendix C. A further impotant feature is the existence of **partitions of unity**. This is a family $\{f_k : X \to [0,1] \subset \mathbb{R}$ of functions subordinate to an open cover $\{U_k\}$ of X, that is

- supp $f_k \subset U_k$,
- family of supports is *locally finite*, that is, any point $a \in X$ has a neighbourhood V which meets only finitely many U_k, so $V \cap U_k = \varnothing$ except for finitely many k,

and such that

- $\sum_k f_k = 0$. (This is the reason calling $\{f_k\}$ a partition of unity. A priori the sum can be infinite but in view of local finiteness, $f_k(p) = 0$ for all but a finite number of k.)

Such a partition of unity exists for any open cover of X, see for instance [Fo, Appendix A].

1. Remark. We will actually assume that X is an *orientable* surface, that is, it does not contain an embedded Moebius band (the one-sided surface you get when you a cylinder by twisting it once, see Figure 1.1). In practice, we can gloss over these topological subtleties.

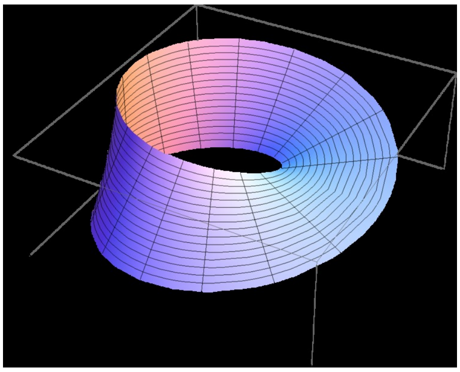

FIGURE 1. A Moebius band

1.1. **Riemann surfaces and holomorphic maps.** First, we define the concept of a Riemann surface before we turn to the functions.

Riemann surfaces.

2. Definition (complex charts and holomorphic atlas. A **complex chart** is a homeomorphism $\varphi : U \subset X \to V \subset \mathbb{C}$ between open sets U and V. Two such charts $\varphi_{1,2} : U_{1,2} \to V_{1,2}$ are said to be **holomorphically compatible** if the maps

$$\varphi_{12} := \varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \to \varphi(U_1 \cap U_2),$$

the so-called **transition functions**, are biholomorphic (see Figure 1.2). We will usually write $U_{12} = U_1 \cap U_2$ for the intersection of two open sets $U_{1,2}$. A **holomorphic atlas** is a system $\mathfrak{A} = \{\varphi_i : U_i \to V_i \mid i \in I\}$ of charts such that

- $\{U_i\}_{i\in I}$ is an open cover of X, i.e. $X = \bigcup_{i\in I} U_i$;
- for any i, $j \in I$, φ_i and φ_j are holomorphically compatible.

Two holomorphic atlases $\mathfrak{A}$ and $\mathfrak{B}$ are **holomorphically equivalent** if any two charts $\varphi \in \mathfrak{A}$ and $\psi \in \mathfrak{B}$ are holomorphically compatible.

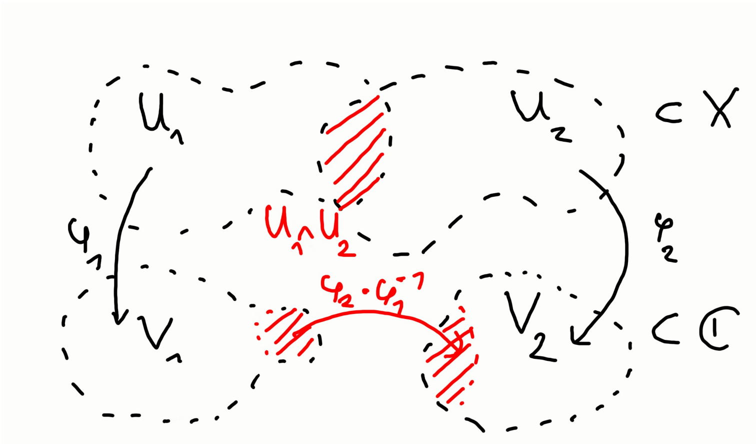

FIGURE 2. Two compatible charts $\varphi_{1,2} : U_{1,2} \subset V_{1,2} \subset \mathbb{C}$

3. Remark. If $\varphi : U \to V$ is a complex chart, then for any open subset $\tilde{U} \subset U$, $\tilde{\varphi} = \varphi|_{\tilde{U}} : \tilde{U} \to \tilde{V} = \varphi(\tilde{U})$ is a complex chart compatible with φ.

It is easily verified that the notion of holomorphic equivalence induces an equivalence relation on atlases.

4. Definition (holomorphic structure and Riemann surface). A **holomorphic structure** on X is an equivalence class of holomorphically equivalent atlases. Every holomorphic structure contains a unique maximal atlas (take the union of all atlases in the equivalence class). Let $\mathfrak{A}$ be such a maximal atlas. Then the pair $(X, \mathfrak{A})$ is called a **Riemann surface**.

5. Remark.

(i) Any oriented surface admits at least one holomorphic structure. This is essentially a consequence of the Uniformisation Theorem (which we will not prove in this course) and the covering theory discussed in Section 1.1.2.

(ii) Since a holomorphic function is necessarily smooth, the identification $\mathbb{C} \cong \mathbb{R}^2$ induces on X the structure of a two dimensional differentiable manifold.

(iii) Radós theorem (see for instance [Fo, Theorem 23.3]) asserts that a Riemann surface is automatically second countable so we could drop this condition from our assumptions on the underlying topological space (the theorem is false however in higher dimensions).

We will usually only specify some atlas of a given equivalence class, not necessarily a maximal one. In particular, if we speak of a chart φ of X, then φ is not necessarily contained in a given atlas, but only compatible (and thus an element of the maximal atlas of the holomorphic structure). More generally, we say that an open set $U \subset X$ is a **coordinate neighbourhood** of X if U is the domain of some compatible chart. If no confusion arises we usually drop any reference to the atlas and denote the Riemann surface by X.

6. Examples. Here are some explicit examples. We will construct further ones by *analytic continuation* below, see Section 1.1.3.

(i) **The complex plane $\mathbb{C}$.** Here, a holomorphic atlas is induced by the equivalence of the atlas $\mathfrak{A} = \{\mathrm{Id} : \mathbb{C} \to \mathbb{C}\}$ which consists solely of the identity. One usually writes $z : \mathbb{C} \to \mathbb{C}$ for this chart and considers z as a holomorphic *coordinate* or *parameter*.

(ii) **Domains.** Let X be a Riemann surface and $Y \subset X$ a domain, i.e. a connected, open subset. We define an atlas of Y by taking all charts $\varphi : U \to \varphi(U)$ of X with $U \subset Y$ (since Y is open, U is open in Y if and only if U is open in X). Hence Y is a Riemann surface, and we will always equip domains of X with this holomorphic structure unless mentioned otherwise.

(iii) **The projective space $\mathbb{P}^1$.** Let

$$\mathbb{P}^1 = \{\text{lines in } \mathbb{C}^2 \text{ through the origin in } \mathbb{C}^2\}.$$

Since any point $(z_0, z_1) \in \mathbb{C}^2$ determines a unique such line, we can identify $\mathbb{P}^1$ with the set $\{[z_0 : z_1] \mid (z_0, z_1) \in \mathbb{C}^2\backslash\{(0,0)\}$. Here, $[z_0 : z_1]$ denotes the equivalence class of $\mathbb{C}^2\backslash\{(0,0)\}$ modulo the action of $\mathbb{C}^*$ by scalar multiplication, i.e. $(z_0, z_1) \sim (w_0, w_1)$ if and only if $z_0 = \lambda w_0$, $z_1 = \lambda w_1$ for some $\lambda \in \mathbb{C}^*$. In particular, we get a projection map $\pi : \mathbb{C}^2\backslash\{(0,0)\} \to \mathbb{P}^1$ which also induces a natural topology on $\mathbb{P}^1$. Namely, a set $U \subset \mathbb{P}^1$ is open if and only if $\pi^{-1}(U)$ is open. In particular, π is continuous. For instance, $U_i = \{[z_0 : z_1] \mid z_i \neq 0\}$ is open, for $\pi^{-1}(U_i) = \{(z_0, z_1) \mid z_i \neq 0\}$. In fact, we get homeomorphisms

$$\varphi_0 : U_0 \to \mathbb{C}, \quad \varphi_0([z_0 : z_1]) = z_1/z_0, \qquad \varphi_1 : U_1 \to \mathbb{C}, \quad \varphi_0([z_0 : z_1]) = z_0/z_1,$$

whose induced transition function is $\varphi_{01} = \varphi_0 \circ \varphi_1^{-1} : \mathbb{C}^* \to \mathbb{C}^*$, $\varphi_{01}(z) = 1/z$ is clearly biholomorphic. This gives $\mathbb{P}^1$ the structure of a Riemann surface. Note in passing that since $\mathbb{P}^1 = \pi(S^3)$ for $S^3 \subset \mathbb{C}^2 \cong \mathbb{R}^4$, $\mathbb{P}^1$ is compact as the image of a compact set under a continuous map.

(iv) **Tori.** Let ω_1, $\omega_2 \in \mathbb{C} \cong \mathbb{R}^2$ be linearly independent over $\mathbb{R}$. Consider the **lattice**

$$\Lambda := \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 = \{n\omega_1 + m\omega_2 \mid n,\, m \in \mathbb{Z}\}$$

spanned by ω_1 and ω_2. We consider the **torus**, the set of equivalence classes

$$T_\Lambda = \mathbb{C}/\Lambda,$$

where two points z, $z' \in \mathbb{C}$ are equivalent if and only if $z - z' \in \Lambda$. Again, we have a projection map $\pi = \pi_\Lambda : \mathbb{C} \to T_\Lambda$ and declare a set U to be open if and only if $\pi^{-1}(U)$ is open in $\mathbb{C}$. To define a complex structure on T_Λ we define an atlas by taking all charts of the following type. We let $V \subset \mathbb{C}$ be an open subset such that no two points in V are equivalent under Λ. In particuar, $\pi|_V : V \to U = \pi(V) \subset T_\Lambda$ is bijective and defines a homeomorphism. Let $\varphi = (\pi|_V)^{-1} : U \to V$. Let us show that any two of these charts, say $\varphi_i : U_i \to V_i$, $i = 1, 2$, are compatible. For the map $\varphi_{12} = \varphi_1 \circ \varphi_2^{-1} : \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$ we have $\pi(\varphi_{12}(z)) = \pi(z)$, that is, $\varphi_{12}(z) - z \in \Lambda$. Since $\varphi_{12}(z) - z$ is continuous and Λ discrete, $\varphi_{12}(z) - z$ is a constant in Λ on any connected component of $\varphi_1(U_1 \cap U_2)$ and thus a translation. In particular, it is holomorphic. Similarly, φ_{12}^{-1} is holomorphic.

7. Remark.

(i) $\mathbb{P}^1$ is also called the *Riemann sphere* for the following reason: If we think of $\mathbb{C}$ as $\mathbb{R}^2$ and let $\infty = [0:1]$, then topologically $\mathbb{P}^1 = \mathbb{R}^2 \cup \{\infty\} \cong S^2$ where the identification of $\mathbb{R}^2 \cup \{\infty\}$ with the 2-sphere is given by stereographic projection (see for instance [Fo, Exercise 1.1]). It follows that S^2 can be given a (unique, see Example 1.18) structure of a Riemann surface. The superscript 1 in $\mathbb{P}^1$ indicates that this is just the first space in the series of higher dimensional projective spaces $\mathbb{P}^n$ obtained by taking the lines through the origin in $\mathbb{C}^{n+1}$.

(ii) One can check that the map $T_\lambda \to S^1 \times S^1$ which takes the point represented by $x\omega_1 + y\omega_2$ to $(\exp(2\pi i x), \exp(2\pi i y))$ is a homeomorphism, and in fact even a diffeomorphism, that is, any two differentiable atlases are equivalent. However, we will see below that there are *inequivalent* holomorphic atlases on T_Λ, that is, T_Λ as a Riemann surface is not uniquely determined (see Example 1.18).

Holomorphic functions. Any good mathematical theory has a notion of (iso-) morphism. In geometry, this is build from the preferred class of functions to which we come next. The resulting morphisms will be studied in the next paragraph.

8. Definition (holomorphic function). Let X be a Riemann surface and $W \subset X$ be an open subset. A function $f : W \to \mathbb{C}$ is called **holomorphic** if for every chart $\varphi : U \to V$ with $U \cap W \neq \emptyset$, the complex function $f \circ \varphi^{-1} : \varphi(U \cap W) \subset \mathbb{C} \to \mathbb{C}$ is holomorphic in the usual sense (cf. Appendix A). The set of holomorphic functions on $W \subset X$ will be denoted by $\mathcal{O}_X(W)$ or $\mathcal{O}(W)$ for short.

9. Remark.

(i) Any constant function is trivially holomorphic, whence a natural inclusion $\mathbb{C} \hookrightarrow \mathcal{O}(W)$. It is straightforward to see that the sum and the product of holomorphic functions are again holomorphic. Since $\mathbb{C} \subset \mathcal{O}(W)$, we see that $\mathcal{O}(W)$ is a $\mathbb{C}$-*algebra*.

(ii) For any domain in $\mathbb{C}$ with its standard holomorphic structure we recover the usual notion of a holomorphic function.

(iii) Any chart $\varphi : U \to V \subset \mathbb{C}$ is holomorphic by the very definition of holomorphic compatibility. Following the notation for $\mathbb{C}$ (cf. (i) of Example 1.6), one also calls $\varphi^{-1} : V \to U$ a local **coordinate** or **uniformising parameter** and writes $z = \varphi^{-1}$. Then a function $f : U \subset X \to \mathbb{C}$ is holomorphic $\Leftrightarrow$ $f(z) : V \to \mathbb{C}$ is holomorphic in the usual sense. Note that almost by definition holomorphicity is a local property and that it is enough to check it for some (not necessarily maximal) atlas of X by compatibility.

(iv) It is actually enough to check holomorphicity for a single atlas $\mathfrak{A}$ which is compatible with the holomorphic structure. Namely, if $f \circ \varphi^{-1} : \varphi(U) \to \mathbb{C}$ is holomorphic for all charts φ in $\mathfrak{A}$, then f is holomorphic. Indeed, let ψ be any complex chart of the Riemann surface. Then (neglecting domains of definition) $f \circ \psi = f \circ \varphi^{-1} \circ \varphi \circ \psi$ The notion was taylor-made for the definition of a holomorphic function.

The classical theorems for holomorphic functionson open sets of $\mathbb{C}$ (cf. Section A) easily generalise to Riemann surfaces, for instance:

10. Theorem (Riemann's Removable Singularities Theorem) [Fo, 1.8]. *Let U be an open subset of a Riemann surface, and let $a \in U$. If $f \in \mathcal{O}(U \backslash \{a\})$ is bounded on $U \Rightarrow f$ can be uniquely extended to a holomorphic function in $\mathcal{O}(U)$.*

Proof. Shrinking U if necessary we can take a chart $\varphi : U \to V$ and consider $f \circ \varphi^{-1} : V \backslash \{\varphi(a)\} \to \mathbb{C}$. This is a bounded holomorphic function on V for f is bounded, hence one can apply the usual version of Riemann's theorem (cf. Appendix A). □

11. Theorem (maximum principle) [Fo, 2.6]. *Let X be a Riemann surface and $f : X \to \mathbb{C}$ be a nonconstant holomorphic function $\Rightarrow |f|$, the absolute value of f, does not have a global maximum.*

Proof. We proceed by contraposition and assume that $M := \sup_{b \in X} |f(b)| < \infty$, that is, $|f|$ attains its global maximum at some point $b \in X$. Hence $b \in S := \{a \in X \mid |f(a)| = M\}$. We have to show that $S = X$. First, S must be closed by continuity of f. Second, S must be also open: Take a chart $\varphi : U \to V$ and an open connected set $U_1 \subset U$ around a so that $f \circ \varphi^{-1}|_{\varphi(U_1)}$ is a holomorphic function with a (global) maximum at $f \circ \varphi^{-1}(a)$. By the classical maximum principle, $f \circ \varphi^{-1}|_{\varphi(U_1)}$ must be constant and thus equal to M, hence $a \in U_1 \in S$. It follows that S is not empty, closed and open and thus equals X for X is connected. □

The maximum principle has also a surprising effect for compact Riemann surfaces.

12. Corollary (holomorphic functions on compact Riemann surfaces) [Fo, 2.8]. *Let X be a compact Riemann surface. Then $\mathcal{O}(X) \cong \mathbb{C}$.*

Proof. Let $f : X \to \mathbb{C}$ be holomorphic so that in particular, f is continuous. Since X is compact it assumes its maximum somewhere. By Theorem 1.11 this is only possible if f is constant. □

13. Corollary (Liouville's theorem) [Fo, 2.10]. *Every bounded function $f : \mathbb{C} \to \mathbb{C}$ is constant.*

Proof. We consider $f \circ \varphi_0 : U_0 \subset \mathbb{P}^1 \to \mathbb{C}$ as a holomorphic function on $\mathbb{P}^1 \backslash \{[0 : 1]\}$. However, since f is bounded, this must be removable by Theorem 1.10 and f extends to a holomorphic function $f : \mathbb{P}^1 \to \mathbb{P}^1$. Hence f is constant by Corollary 1.12. □

Holomorphic maps. Now we can define the notion of a holomorphic map. Taking these as *morphisms* we can actually define a *category*, but we will not pursue this viewpoint further.

14. Definition (holomorphic map). Suppose X and Y are Riemann surfaces. A continuous map $F : X \to Y$ is called **holomorphic**, if for every pair of charts

$\varphi_1 : U_1 \to V_1$ and $\varphi_2 : U_2 \to V_2$ on X and Y respectively with $F(U_1) \subset U_2$, the function

$$\varphi_2 \circ F \circ \varphi_1^{-1} : V_1 \subset \mathbb{C} \to V_2 \subset \mathbb{C}$$

is holomorphic in the usual sense. A map $F : X \to Y$ is **biholomorphic** if there exists a holomorphic map $G : Y \to X$ such that $F \circ G = \mathrm{Id}_Y$ and $G \circ F = \mathrm{Id}_X$, that is, F is bijective and has a holomorphic inverse $F^{-1} = G$. If a biholomorphic map $X \to Y$ exists we say that X and Y are **isomorphic**.

15. Remark.

(i) The composition of two holomorphic maps is again holomorphic.
(ii) If $Y = \mathbb{C}$, then holomorphic maps are just holomorphic functions in the sense of Definition 1.8.
(iii) Two holomorphic atlases $\mathfrak{A}$ and $\mathfrak{A}'$ on X are equivalent if and only if the identity $\mathrm{Id} : (X, \mathfrak{A}) \to (X, \mathfrak{A}')$ is a biholomorphic map.

More generally, a *continuous map* $F : X \to Y$ is holomorphic if and only if for every chart $\varphi : U \subset Y \to V$, the restricted function $F^*\varphi := \varphi \circ F|_{F^{-1}(U)} : F^{-1}(U) \to V \subset \mathbb{C}$ is holomorphic.

16. Proposition [Gu, Lemma 2]**.** $F : X \to Y$ *is holomorphic* $\Leftrightarrow$ *for any open subset* $U \subset Y$*, we have* $F^*f \in \mathcal{O}_X(F^{-1}(U))$*, that is, we get an induced map*

$$F^*_U : \mathcal{O}_Y(U) \to \mathcal{O}_X(F^{-1}(U)).$$

Proof. If F is holomorphic, then clearly $F^*_U f \in \mathcal{O}_X(F^{-1}(U))$ for all $f \in \mathcal{O}_Y(U)$. For the converse we need to check that F is holomorphic near any $p \in X$. Take coordinates φ and ψ around p and $F(p)$ respectively, say on $U \subset X$ and $V \subset Y$ with $F^{-1}(V) \subset U$. In particular, $\psi \in \mathcal{O}_Y(V)$. Hence $F^*\psi \in \mathcal{O}_X(F^{-1}(V))$ by assumption so that $F^*\psi \circ \varphi^{-1} = \psi \circ F \circ \varphi$ is a holomorphic function in the ordinary sense. ☐

17. Remark. It is easily checked that F^* is a ring morphism. If $F : X \to Y$ and $G : Y \to Z$ are holomorphic maps, then $(G \circ F)^* = F^* \circ G^*$.

18. Examples.

(i) The famous *Uniformisation Theorem* asserts that any simply-connected Riemann surface is isomorphic to either of the following ones: $\mathbb{P}^1$, $\mathbb{C}$ or a domain strictly contained in $\mathbb{C}$ (any two such domains are isomorphic by Riemann's mapping theorem). In particular, there exists only one compact simply-connected Riemann surface. Put differently, any oriented compact surface of genus zero has precisely one holomorphic structure up to biholomorphic maps (for instance induced by linear transformations of the form $A[z_0 : z_1] = [Az_0 : Az_1]$ for $A \in \mathrm{GL}(2, \mathbb{C})$).
(ii) Next let us consider a compact Riemann surfaces of genus 1, i.e. tori. Let $\Lambda = \{m_1\omega_1 + m_2\omega_2 \mid m_i \in \mathbb{Z}\}$ and $\Lambda' = \{m_1\omega_1' + m_2\omega_2' \mid m_i \in \mathbb{Z}\}$ be two lattices in $\mathbb{C}$. Let $T = T_\Lambda = \mathbb{C}/\Lambda$ and $T' = T_{\Lambda'} = \mathbb{C}/\Lambda'$ be the corresponding complex tori. If T and T' are isomorphic via an isomorphism $F : T \to T'$, we can lift $F \circ \pi_\Lambda : \mathbb{C} \to T_{\Lambda'}$ by standard covering space theory (cf. Appendix B, in particular Theorem B.15) to a periodic holomorphic map $\tilde{F} : \mathbb{C} \to \mathbb{C}$ which

satisfies $\pi_{\Lambda'} \circ \tilde{F} = F \circ \pi_\Lambda$. Since $\pi_{\Lambda'}(\tilde{F}(z+\lambda)) = \tilde{F}(\pi_\Lambda(z+\lambda)) = \pi_{\Lambda'}(\tilde{F}(z))$ for all $\lambda \in \Lambda$ we have $\tilde{F}(z+\lambda) - \tilde{F}(z) \in \Lambda'$, i.e

$$\tilde{F}(z+\omega_i) = \tilde{F}(z) + a_{i1}\omega_1' + a_{i2}\omega_2' \tag{1}$$

for some integers $a_{ij} \in \mathbb{Z}$ such that $a_{11}a_{22} - a_{12}a_{21} = \pm 1$. The latter condition stems from the fact that the inverse $\tilde{F}^{-1}$ must satisfy a similar relation, that is, $\tilde{F}^{-1}(z+\omega_i') = \tilde{F}^{-1}(z) + b_{i1}\omega_1 + b_{i2}\omega_2$ for $b_{ij} \in \mathbb{Z}$. Moreover, the matrices satisfy $(b_{ij}) = (a_{ij})^{-1}$. By Cramer's rule happens it follows that $\det(a_{ij}) = \pm 1$. By interchanging the order of $\omega_{1,2}'$ we may assume that $\det(a_{ij}) = 1$, that is, $(a_{ij}) \in \mathrm{SL}(2,\mathbb{Z})$. Since $\tilde{F}(z+\lambda) - \tilde{F}(z) \in \Lambda'$ is constant in z, differentiating in z yields $\tilde{F}'(z) = \tilde{F}'(z+\lambda)$. Hence $\tilde{F}'$ is invariant under Λ and thus decends to a function on the torus $T_\Lambda \to \mathbb{C}$. By Theorem 1.12 F' must be constant: $\tilde{F}'(z) \equiv c \in \mathbb{C}$. Hence $F(z) = cz + d$ for a further constant $d \in \mathbb{C}$. We can get rid of d by compounding $\tilde{F}$ with the biholomorphic map induced by the translation $\tau : T_{\Lambda'} \to T_{\Lambda'}$, $\tau([z]) = [z-d]$ which lifts to the translation $\tilde{\tau} : \mathbb{C} \to \mathbb{C}$, $\tilde{\tau}(z) = z - d$. So up to a translation, $\tilde{F}(z) = cz$. Then Equation (1) implies

$$c\omega_1 = a_{11}\omega_1' + a_{12}\omega_2', \quad c\omega_2 = a_{21}\omega_1' + a_{22}\omega_2'. \tag{2}$$

If we consider the ratios $\omega = \omega_1/\omega_2$ and $\omega' = \omega_1'/\omega_2'$, the latter relation gives

$$\omega = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \omega' := \frac{a_{11}\omega' + a_{12}}{a_{21}\omega' + a_{22}}. \tag{3}$$

Conversely, if (3) holds, then there exists a complex constant $c \neq 0$ such that (2) $\tilde{F}(z) := cz$ satisfies (1) and thus descends to an isomorphism $F : T \to T'$. Hence T and T' are isomorphic $\Leftrightarrow$ $\omega = A\omega'$ for $A \in \mathrm{SL}(2,\mathbb{Z})$ and $A\omega'$ defined as in (3).

19. Remark. In particular, two nonisomorphic holomorphic structures can give rise to the same differentiable structure as the previous example of nonisomorphic tori shows (as observed in Example 1.6 (iv), they are diffeomorphic to $S^1 \times S^1$).

Next we prove some elementary properties of holomorphic maps.

20. Theorem (Identity Theorem) [Fo, 1.11]. *Let $F_{1,2} : X \to Y$ be two holomorphic maps between two Riemann surfaces X and Y. If there exists a set $S \subset X$ with a limit point (e.g. S open) such that $F_1|_S = F_2|_S$, then $F_1 \equiv F_2$.*

Proof. Let R be the set of all points $a \in X$ which have an open neighbourhood U such that $F_1|_U = F_2|_U$.

Step 1. *R is not empty.* Indeed, consider charts $\varphi_1 : U_1 \to V_1$ and $\varphi_2 : U_2 \to V_2$ with U_1 connected, $a \in U_1$ and $F_i(U_1) \subset U_2$. Since $\varphi_1(S \cap U_1)$ contains a limit point, $\varphi_2 \circ F_1 \circ \varphi_1^{-1} = \varphi_2 \circ F_2 \circ \varphi_1^{-1} : V_1 \to \mathbb{C}$ by the usual identity theorem for holomorphic functions, cf. Corollary A.11. Hence $F_1|_{U_1} = F_2|_{U_2}$, so $a \in R$.

Step 2. *R is open.* This follows by design.

Step 3. *R is closed.* Let $b \in \partial R$ be a boundary point of R. Then $F_1(b) = F_2(b)$ by continuity. Let U be an open neighbourhood of b. Take a chart $\varphi : U \to V$ with U connected and $b \in U$. Since b is a boundary point of R, $U \cap R \neq \varnothing$. Arguing as in the first step we see that $F_1|_U = F_2|_U$, whence $b \in R$.

The result now follows from the connectivity of X. □

21. Corollary. *If U is connected, then $\mathcal{O}(U)$ is an integral domain.*

Proof. Indeed, assume that $f \cdot g \equiv 0$ for $f,\ g \in \mathcal{O}(U)$. Then either g or f must vanish on an open subset of U, and hence on all of U by the Identity Theorem (applied to the case $Y = \mathbb{C}$). $\square$

Recall that a subset S of a topological space X is called **discrete** if every point $a \in S$ has an open neighbourhood U in X such that $U \cap S = \{a\}$. From the Identity Theorem 1.20 we immediately deduce the

22. Corollary [Fo, 4.2]**.** *Let $F : X \to Y$ be a nonconstant holomorphic map . Then F is discrete, i.e. F has discrete fibres.*

Proof. Otherwise, there exists $b \in Y$ such that the set $S = \{a \in X \mid F(a) = b\}$ has an accumulation point. But then $F \equiv b$, i.e. F would be constant. $\square$

23. Example. Consider the holomorphic projection $\pi_\Lambda : \mathbb{C} \to T_\Lambda$. The fibres can be identified with translated copies of Λ which is discrete in $\mathbb{C}$.

Next we are proving further elementary properties of holomorphic maps based on the following *local classification*:

24. Theorem (local normal form theorem for holomorphic maps) [Fo, 2.1]**.** *Let X and Y be Riemann surfaces, and $F : X \to Y$ is a nonconstant holomorphic map. Suppose $a \in X$ and $b := F(a)$. Then there exists an integer $k \geqslant 1$ and charts $\varphi : U \to V$ on X and $\psi : U' \to V'$ on Y such that*

(i) *$a \in U$, $\varphi(a) = 0$ and $b \in U'$, $\psi(b) = 0$;*
(ii) *$F(U) \subset U'$;*
(iii) *the map $\psi \circ F \circ \varphi^{-1} : V \to V'$ is the assignement $z \mapsto z^k$.*

Proof. It is clear that we can find charts satisfying the first two properties, i.e. $F_1 = \psi \circ F \circ \varphi^{-1}$ satisfies $F_1(0) = 0$. Hence there exists a $k \geqslant 1$ such that $F_1(z) = z^k g(z)$ with $g(0) \neq 0$. On a (simply-connected) neighbourhood of 0 we can thus take the k-th root of g, i.e. there exists a holomorphic function h defined near 0 such that $h^k(z) = g(z)$. If we let $\varphi_1(z) = zh(z)$ then φ_1 is biholomorphic near 0 onto its image. Further, $\varphi(z)^k = F(z)$, whence $F_1 \circ \varphi^{-1}(z) = F_1(\varphi^{-1}(z)) = z^k$ as desired. $\square$

25. Remark. There are two cases to consider: Either $k = 1$ so that F is locally injective, or $k \geqslant 2$ and we have a *branch point*, see Definition 1.36. In particular, k does not depend on the choice of charts as these are injective. We therefore have an intrinsic interpretation of the number k, namely as the *order of ramification*, cf. Example 1.38 (ii). Namely, for any open neighbourhood U of a, there exists an open neighbourhood $U_0 \subset U$ such that $f^{-1}(y) \cap U_0$ has precisely k elements if $y \neq b$. One calls k also the **multiplicity** of a for which we write $k = \nu(F, a)$.

26. Example. Let $p(z) = z^k + \sum_{i=0}^n c_i z^i \in \mathbb{C}[z]$ be a nonconstant complex polynomial considered as a holomorphic map $P : \mathbb{P}^1 \to \mathbb{P}^1$ by setting $P([1 : z]) =$

$[1:p(z)]$ and $f([0:1]) = [0:1]$. Using the chart φ_1 near $[0:1]$ (cf. Example 1.6 (iii)) we have

$$\varphi_1 \circ P \circ \varphi_1^{-1}(w) = \varphi_1\big(P([w:1])\big) = \begin{cases} 0, & w = 0 \\ \frac{w^n}{\sum_{k=0}^n a_k w^{n-k}}, & w \neq 0 \end{cases}$$

Since $1/\sum a_k w^{n-k} \neq 0$ for w sufficiently close to 0, P is indeed holomorphic as it is bounded near $w = 0$, and we can argue as for Theorem 1.24 and find a new chart around $\infty = [0:1]$ so that P expressed in these coordinates is of the form w^n. Hence $\nu(P, \infty) = n$.

It follows that as far as local properties are concerned, we may *always* assume that modulo charts, $F : X \to Y$ is locally of the form $F(z) = z^k$. This implies immediately the following

27. Corollary [Fo, 2.4, 2.7]**.** *Let $F : X \to Y$ be a nonconstant holomorphic map. Then*

(i) *F is open;*
(ii) *if X is compact, then F is surjective. In particular, Y is compact.*

Proof. The map $z \mapsto z^k$ is clearly open. In particular $F(X) \subset Y$ is open. But if X is compact, then $F(X)$ is compact and thus closed. Hence $F(X) = Y$ by connectivity of Y. □

28. Corollary [Fo, 2.5]**.** *Let $F : X \to Y$ be an injective holomorphic map. Then $F : X \to F(X)$ is biholomorphic.*

Proof. We necessarily have $k = 1$ for F is injective. □

Meromorphic functions. Next we generalise the concept of meromorphic functions to Riemann surfaces. While in a usual course on complex analysis, these are treated as a generalisation of a holomorphic function, on Riemann surfaces we can see them as a special class of holomorphic maps (cf. Proposition 1.32) which is why we treat them now.

29. Definition. Let X be a Riemann surface and $U \subset X$ be open. We call $f : U \dashrightarrow \mathbb{C}$ a **meromorphic function on** U if $f : U^\times \to \mathbb{C}$ is a holomorphic function defined on some open subset $U^\times \subset U$ such that

(i) $U \backslash U^\times$ contains only isolated points;
(ii) for every point $a \in U \backslash U^\times$ one has

$$\lim_{z \to a} |f(z)| = \infty.$$

The points of $U \backslash U^\times$ are the **pôles** of f. The set of all meromorphic functions on U is denoted by $\mathcal{M}_X(U)$ or $\mathcal{M}(U)$ for short.

30. Remark.

(i) Using a chart $\varphi : U \to \mathbb{C}$ one immediately sees that $f \in \mathcal{M}_X(U)$ gives rise to a meromorphic function $f \circ \varphi^{-1} : \varphi(U) \dashrightarrow \mathbb{C}$ in the usual sense (cf. Appendix A.26). In particular, a is a pôle $\Leftrightarrow$ there exists a minimal $m \geqslant 1$ such that $z^m f \circ \varphi^{-1}(z)$ is bounded near a for any chart near a with $\varphi(a) = 0$ (this is indeed independent of the chart as φ is biholomorphic). We call m the **order** of the pôle. Therefore we can locally develop $f \circ \varphi$ into a Laurent series $\sum_{k \geqslant \nu} a_k z^k$ where $a_k \in \mathbb{C}$ and $\nu \in \mathbb{Z}$ is the order of $\varphi^{-1}(0)$.

(ii) $\mathcal{M}(U)$ has the natural structure of a $\mathbb{C}$-algebra (and in fact of a field, see Corollary 1.33). Indeed, for $f, g \in \mathcal{M}(U)$ we define $f + g$ and $f \cdot g$ by taking first their sum and product on the open subset $U^\times$ where they are both simultanuously holomorphic and then extending over any removable singularity (cf. Theorem 1.10). Note that pôles can indeed cancel (consider z and $1/z$ in $\mathcal{M}_{\mathbb{C}}(\mathbb{C})$).

31. Examples.

(i) **Holomorphic functions:** Any holomorphic function is obviously meromorphic with empty pôle set, i.e. $\mathcal{O}(U) \subset \mathcal{M}(U)$.

(ii) **Polynomials on $\mathbb{P}^1$:** Consider again Example 1.26 where the polynomial $p(z) = z^k + \sum_{i=0}^n c_i z^i \in \mathbb{C}[z]$ gave rise to the holomorphic map $P : \mathbb{P}^1 \to \mathbb{P}^1$ by setting $P([1 : z]) = [1 : p(z)]$ and $P([0 : 1]) = [0 : 1]$. The restriction $P|_{U_0}$ takes then values in $\mathbb{C}$ and can be thus regarded as a holomorphic function defined on $\mathbb{P}^1 \backslash \{\infty\}$. Read in the chart φ_1 with local coordinate $w = 1/z$ over $U_0 \cap U_1$, we have $P(w) = \sum_{j=0}^n a_i w^{-i}$ which clearly has a pole at 0 of order n.

Examples 1..26 and 1.31 (ii) show that polynomials $P : \mathbb{C} \to \mathbb{C}$ can be either considered as holomorphic maps $\mathbb{P}^1 \to \mathbb{P}^1$ or as meromorphic functions $\mathbb{P}^1 \dashrightarrow \mathbb{C}$. The next proposition shows that global meromorphic functions on a Riemann surface $X \dashrightarrow \mathbb{C}$ correspond to holomorphic maps $X \to \mathbb{P}^1$ which emphasises once more the special rôle played by $\mathbb{P}^1$. Subsequently, we will tacitely identify these two viewpoints and think of meromorphic functions as holomorphic functions $X \to \mathbb{P}^1$ and vice versa.

32. Proposition [Fo, 1.15]**.** *Let X be a Riemann surface and $f \in \mathcal{M}(X)$. For each pôle p of f, define $F(p) := \infty$, and $F = f$ on $X^\times$. Then $F : X \to \mathbb{P}^1$ is a holomorphic map. Conversely, if $F : X \to \mathbb{P}^1$ is a holomorphic map, then F is either identically equal to $\infty = [0 : 1]$ or else $F^{-1}(\infty)$ consists of isolated points and $f : U^\times := X \backslash F^{-1}(\infty) \to \mathbb{C}$ induces a meromorphic function $f : X \dashrightarrow \mathbb{C}$.*

Proof. For $f \in \mathcal{M}(X)$ let $P(f)$ be the set of pôles of f. We define a continuous extension of $F : X^\times = X \backslash P(f) \to \mathbb{C}$ to $f : X \to \mathbb{P}^1$ by setting $f(p) = [0 : 1] = \infty$. If $\psi : V \to \mathbb{C}$ is a chart such that $U \cap P(f) = \{p\}$ such that $F(V) \subset U_1$ in $\mathbb{P}^1$ $\Rightarrow \varphi_1 \circ F \circ \psi^{-1} : \psi(U) \to \mathbb{C}$ is continuous. Since and $\psi : V \to \mathbb{C}$ is a chart of $\mathbb{P}^1$ then $\varphi_1 \circ F \circ \psi$ is continuous, so that the singularity in $\psi(p)$ is removable by Theorem 1.10.

The converse follows from the Identity Theorem 1.20. □

33. Corollary. *The Identity Theorem 1.20 holds for meromorphic functions regarded as holomorphic maps. In particular, the set of zeroes and poles of a meromorphic functions is discrete. It follows that any meromorphic function which is* *not identically zero can be inverted so that*

$$K_X := \mathcal{M}(X)$$

is a field. We call K_X the **function field** *of X.*

Unlike for holomorphic functions there *are* nonconstant global meromorphic functions on compact Riemann surfaces:

34. Example (the function field of $\mathbb{P}^1$) [Fo, 2.9]**.** We have

$$K_{\mathbb{P}^1} = \mathbb{C}(T),$$

that is, any global meromorphic function can be written as the quotient of two polynomials so that f is rational. Indeed, let $f \in K_{\mathbb{P}^1}$. Then f has finitely many poles $a_1, \ldots, a_n$, and by passing to $1/f$ if necessary, we may assume that all of these poles live in U_0. Fixing a coordinate z we have prinicipal parts $h_k = \sum_{j=-\nu_k}^{-1} c_{\nu_k j}(z-a_k)^j$ of the corresponding Laurent series which we can holomorphically extend to all of $\mathbb{P}^1$ as they are bounded if $z \to \infty$ (use the Removable Singularity Theorem; cf. also Example 1.31). Hence $c = f - \sum h_k$ must be holomorphic on $\mathbb{P}^1$ and thus be constant. It follows that $f = c + \sum h_k$ is a rational function. A generator is given by the meromorphic function $[1 : z] \to z$ which is why we usually write $K_{\mathbb{P}^1} = \mathbb{C}(z)$

35. Examples (the function field of T_Λ and doubly periodic functions). Next we consider genus 1 surfaces. Nontrivial global meromorphic functions on complex tori arise for instance from doubly periopdic functions: Suppose as above that ω_1, $\omega_2 \in \mathbb{C}$ are linearly independent over $\mathbb{R}$ and let $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ be the induced lattice. A meromorphic function $F : \mathbb{C} \dashrightarrow \mathbb{C}$ is called **doubly periodic** if $F(z + \omega_1) = F(z) = F(z + \omega_2)$ or equivalently, if $F(z + \omega) = F(z)$ for all $\omega \in \Lambda$. In particular, F descends to a meromorphic map $F : T_\Lambda \to \mathbb{P}^1$. For instance, the **Weierstrass $\wp$-function with respect to Λ** is defined by

$$\wp_\Lambda(z) = \frac{1}{z^2} + \sum_{\omega \in \Gamma \backslash 0} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right).$$

Conversely, any meromorphic function $T_\Lambda \dashrightarrow \mathbb{C}$ gives rise to a doubly periodic function so we can freely identify these two concepts. From Corollaries 1.12 and 1.27 we immediately deduce that any *holomorphic* doubly periodic function $\mathbb{C} \to \mathbb{C}$ must be constant. Moreover, any nonconstant meromorphic doubly periodic function must attain any complex value for it induces a holomorphic map $f : T_\Lambda \to \mathbb{P}^1$ which by Corollary 1.27 (ii) is surjective.

This function will be further investigated in the exercise sheets. Furthermore, it will be shown that

$$\begin{aligned} K_{T_\Lambda} &\cong \mathbb{C}(z)[X]/(X^2 - 4(z - \wp(\omega_1/2))(z - \wp(\omega_2/2))(z - \wp((\omega_1 + \omega_2)/2))) \\ . &\cong K_{\mathbb{P}^1}[X]/(X^2 - 4(z - \wp(\omega_1/2))(z - \wp(\omega_2/2))(z - \wp((\omega_1 + \omega_2)/2))) \end{aligned}$$

(this does indeed only depend on the lattice, and not on the basis ω_i). In particular, $K_{\mathbb{P}^1} \subset K_{T_\Lambda}$ is a finite field extension.

1.2. **Branch points.** We have now introduced the basic objects of study of this course, namely Riemann surfaces and holomorphic maps between them. Next we investigate the structure of holomorphic maps in finer detail.

Since the fibres of holomorphic maps are discrete we can make a rough a subdivision into *branched* and *unbranched* holomorphic maps.

36. Definition. Let $F : X \to Y$ be a nonconstant holomorphic map. A point $a \in X$ is called a **ramification point** of F, if there is no neighbourhood U of a such that $F|_U$ is injective. If F has branch points, then F is called **branched** or **ramified**, and **unbranched** or **unramified** else.

37. Remark. There does not seem to be a universal agreement on the distinction between ramification and branch points in the literature so be careful when using other texts.

38. Examples.

(i) Let $k \geqslant 2$ be a natural number and let $P_k : \mathbb{C} \to \mathbb{C}$ be the map $P_k(z) = z^k$. Then $0 \in \mathbb{C}$ is a ramification point as well as a branch point, while the map $P_k : \mathbb{C}^* \to \mathbb{C}$ is unbranched.

(ii) By the Normal Form Theorem 1.24, any nonconstant holomorphic map $F : X \to Y$ looks locally like P_k. Hence $a \in X$ is a ramification point precisely if its multiplicity is $\geqslant 2$.

(iii) The mapping $\exp : \mathbb{C} \to \mathbb{C}^*$ is an unbranched holomorphic map, for exp is injective on any domain which does not contain two points differing by an integral multiple of $2\pi i$.

(iv) The canonical projection $\pi : C \to T_\Lambda$ onto the torus defined by the lattice Λ is unbranched, for π is a local homeomorphism.

Thus there are three kinds of holomorphic maps:

- constant maps;
- unbranched maps;
- branched maps.

Of course, there is not much to say about constant maps. We first analyse the unbranched case.

Covering maps. The first important property of unbranched maps is the following characterisation which generalises Example 1.38 (iv).

39. Proposition (unbranched maps are local homeomorphisms) [Fo, 4.4]**.** *A nonconstant holomorphic map $F : X \to Y$ has no ramification points if and only if F is a local homeomorphism, i.e. every point $a \in X$ has an open neighbourhood which under F is mapped homeomorphically onto an open neighbourhood of $F(a)$.*

Proof. $\Rightarrow$) Suppose $F : X \to Y$ has no ramification points. Hence for $a \in X$ there exists an open neighbourhood U such that $F|_U$ is injective. Since F is open as a nonconstant holomorphic map, $F|_U$ is a homeomorphism between U and $F(U)$.

$\Leftarrow$) Suppose $F : X \to Y$ is a local homeomorphism. Then any point $a \in U$ admits by definition an open neighbourhood U such that $F|_U$ is injective. $\square$

A convere to the previous proposition is this.

40. Proposition [Fo, 4.6]**.** *Let X be a Haudorff space and Y be a Riemann surface. If $F : X \to Y$ is a local homeomorphism $\Rightarrow$ there exists a unique complex structure on Y such that F is an unbranched holomorphic map.*

Proof. We proceed in two steps.

Step 1. *Existence.* We let $\mathfrak{A}$ be the family of charts constructed as follows. For a complex chart $\varphi' : U' \subset Y \to V$ around a point in the image of F we let $U \subset X$ be such that $F(U) \subset U'$ and $F|_U$ is a homeomorphism onto its open image. We then define the complex chart $\varphi = \varphi' \circ F : U \to \varphi'(F(U))$. It is clear that these charts are compatible (the F just cancels), and the coordinate neighbourhoods cover X. Furthermore, F (trivially) becomes locally bihiolomorphic and is thus holomorphic.

Step 2. *Uniqueness.* Assume that there is another atlas $\mathfrak{A}'$ such that $F : (X, \mathfrak{A}') \to Y$ is holomorphic. Then $\mathrm{Id} : (X, \mathfrak{A}) \to (X, \mathfrak{A}')$ is biholomorphic since locally, $\mathrm{Id}(a) = (p|_U)^{-1} \circ p(a)$ for a suitable open set U.

□

41. Holomorphic covering maps. A special and very important class of local homeomorphisms is given by covering maps (see Definition B.16 and Appendix B for further information). We now investigate the relation of holomorphic unbranched maps with covering maps. Let us start with some basic observations:

(i) If $\pi : X \to Y$ is a covering map and Y is a Riemann surface, we obtain a unique Riemann surface structure on X so that π becomes an unbranched holomorphic map by Proposition 1.40.
(ii) A proper unbranched holomorphic map $\pi : X \to Y$ is a covering map with finite fibres by Proposition B.22.
(iii) Any Deck transformations of a holomorphic covering is necessarily holomorphic. Indeed, we have the following: *Assume that X, Y and Z are Riemann surfaces, and that $\pi : X \to Y$ is an unbranched holomorphic map. Then every lift of a holomorphic map $F : Z \to Y$ to X is holomorphic.* This can be shown by restricting to a neighbourhood $U \subset X$ such that $\pi|_U$ is biholomorphic onto its image [Fo, Theorem 4.9]. Since a Deck transformation is a lift of the map $F = \pi : Z = X \to Y$, it is necessarily holomorphic.

For holomorphic covering maps there are two cases to consider, namely whether the fibres are finite or not. We assume finiteness first which implies that the map $F : X \to Y$ is proper. In particular, F is *closed*, that is, it maps closed sets in X to closed sets in Y.

The set of ramification points $\mathcal{R}$ is a closed discrete subset of X as follows from the local normal form theorem 1.24. Since F is proper, $\mathcal{B} = F(\mathcal{R})$, the set of branch points, is also closed and discrete. It follows that $F|_{X \backslash \mathcal{R}}$ is a holomorphic covering map onto $Y \backslash \mathcal{B}$ with a well-defined number of sheets by Proposition B.14. This means that every value $b \in Y \backslash \mathcal{B}$ of F is taken exactly n times. We also say that b has *multiplicity* n. In order to extend that notion over all of Y, we define the multiplicity $\mu(F, b)$ of any point $b \in X$ by

$$\mu(F, b) = \sum_{a \in \pi^{-1}(b)} \nu(F, a)$$

where $\nu(F, a)$ is the multiplicity of $a \in X$. For ramification points one also considers the **ramification index** which is $\rho(F, a) := \nu(F, a) - 1$. In particular, $\mathcal{R}(F) = \{a \in$

$X \mid \rho(F,a) > 0\}$. A covering is **simply ramified** or has a **simple ramification point** at a if $\rho(F,a) = 1$, has a **double ramification point** if $\rho(F,a) = 2$ and so on.

42. Remark. Consider an n-sheeted branched holomorphic covering map $F : X \to Y$ of two compact Riemann surfaces of genus g and g', respectively. A priori, the ramification index of any ramification point can be any number between 1 and $n-1$. The total sum $\sum_{a\in\mathcal{R}} \rho(F,a)$, however, is topologically determined. Indeed, we have the **Riemann-Hurwitz formula** [Fo, 17.14]

$$2(g-1) = 2n(g'-1) + \sum_{a\in\mathcal{R}} \rho(F,a).$$

In particular, an unbranched holomorphic covering map must have $g-1/g'-1$ sheets. This formula easily follows from Euler's formula for the Euler characteristic $\chi(X) = 2(g-1) = V - E + F$ where E, K and F denotes the total number of vertices, edges, and faces of a triangulation (taking $-E$ ensures that $\chi(X)$ is indeed independent of the chosen triangulation). See also Appendix C for a recap on the topology of surfaces.

43. Proposition [Fo, 4.24]**.** *If F is a proper nonconstant holomorphic map $\Rightarrow$ $\mu(F,b)$ is constant on Y. We call $\mu(F,b)$ the* **number of sheets**.

Proof. If we take out the ramification points of X, then F restricts to a covering map of say n sheets. Let $b \in \mathcal{B}$ and $F^{-1}(b) = \{a_1,\dots,a_r\}$. Now for all i there exists disjoint neighbourhoods U_i of a_i, and neighbourhoods V_i of b, such that $F^{-1}(c)\cap U_i$ has precisely $\nu(F,a_i)$ elements for $c \in V_i\backslash\{b\}$. Since F is a covering over $V\backslash\{b\}$, the cardinality of $F^{-1}(c)$ for $c \in V\backslash\{b\}$ is n, whence the $\sum_i \nu(F,a_i) = n$ $\square$

44. Example. Let $p(z,w) = \sum_{i=0}^n f_i(w)z^i \in \mathcal{O}(\mathbb{C})[z]$ a polynomial with coefficients in $\mathcal{O}(\mathbb{C})$. We let $X = \mathcal{Z}(p) = \{(z,w) \in \mathbb{C}^2 \mid f(z,w) = 0\}$, $Y = \mathbb{C}$ and $F : X \to Y$, $F(z,w) = w$ projection on the second factor. Under mild conditions (namely $\partial_z p(z_0,w_0)$ or $\partial_w p(z_0,w_0) \neq 0$ for $(z_0,w_0) \in X$) the implicit function theorem for holomorphic functions (cf. Remark A.9 and [GuRo, Theorem I.B.4]) implies that X is a Riemann surface. Generically, the polynomial has n distinct roots, and F defines a covering map. It branches over multiple zeroes, see Figure 1.3.

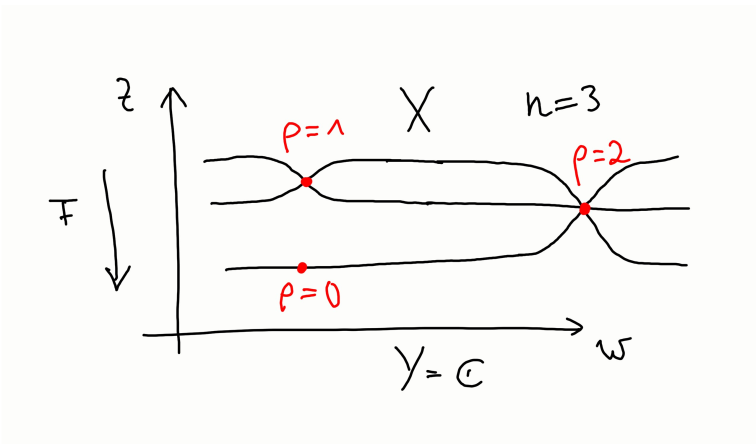

FIGURE 3. The covering map defined by $p \in \mathcal{O}(\mathbb{C})[z]$ of degree 3

45. Corollary [Fo, 4.25]. *A nonconstant meromorphic function over a compact Riemann surface has as many poles as zeroes (counted with multiplicities).*

Proof. Consider the meromorphic function as a holomorphic map $X \to \mathbb{P}^1$. Since it is proper, $\mu(F,\infty) = \mu(F,0)$. □

Summarising, a holomorphic map between compact Riemann surfaces $F : X \to Y$ is either constant or a (branched) covering map with finite fibres. Next we study local normal forms for holomorphic (branched) coverings. In the sequel, we let $D = \{z \in \mathbb{C} \mid |z| < 1\}$, $D^\times = D\backslash\{0\}$ and $\mathbf{H} = \{z \in \mathbb{C} \mid \mathrm{Im}\,|z| > 0\}$. Recall from Example B.17 that $p_k : D^\times \to D^\times$, $p_k(z) = z^k$ and $\exp(i\cdot) : \mathbf{H} \to D^\times$ are covering maps.

46. Theorem (local classification of holomorphic covering maps) [Fo, 5.10]. *Let $F : X \to D^\times$ be an unbranched holomorphic covering map. Then*

(i) *If the covering has an infinite number number of sheets $\Rightarrow$ there exists a biholomorphic mapping $\Phi : X \to \mathbf{H}$ such that the diagramm*

$$\begin{tikzcd} X \arrow[r, "\Phi"] \arrow[d, "F"] & \mathbf{H} \arrow[dl, "\exp(i\cdot)"] \\ D^\times & \end{tikzcd}$$

commutes.

(ii) *If the covering is k-sheeted $\Rightarrow$ there exists a biholomorphic mapping $\Phi : X \to D^\times$ such that the diagramm*

$$\begin{tikzcd} X \arrow[r, "\Phi"] \arrow[d, "F"] & D^\times \arrow[dl, "p_k"] \\ D^\times & \end{tikzcd}$$

commutes.

Proof. This follows directly from Proposition B.32 for Deck $(X/D^\times)$ must be a subgroup of $\pi_1(D^\times) \cong \mathbb{Z}$. The holomorphicity of Φ follows in the same way as for Deck transformations in (iii) of Paragraph 1.41. □

47. Corollary [Fo, 5.11]. *Let $F : X \to D$ be a proper non-constant holomorphic covering map such that F restricted to $X^\times := F^{-1}(D^\times) \to D^\times$ is an unbranched covering map $\Rightarrow$ there exist $k \in \mathbb{N}$ and a biholomorphic mapping $\Phi : X \to D$ such that the diagramm*

$$\begin{tikzcd} X \arrow[r, "\Phi"] \arrow[d, "F"] & D \arrow[dl, "p_k"] \\ D & \end{tikzcd}$$

commutes.

Proof. By the previous theorem, the restriction of F to $X^\times$ factorises via a holomorphic map $\Phi : X^\times \to D^\times$ into $F = p_k \circ \Phi$. If we can extend Φ to all of X we are done. For this we show that $F^{-1}(0)$ consists of a single point a so that we obtain a continuous, hence holomorphic extension by $\Phi(a) = 0$. Indeed, assume that $F^{-1}(0)$ consisted of n points $a_1, \ldots, a_n$ with $n \geqslant 2$. Since they are isolated we have $F^{-1}(D_\epsilon) \subset V_1 \cup \ldots \cup V_n$ for a small neighbourhood D_ϵ around 0 and disjoint open neigbourhoods V_i of a_i. Let $D_\epsilon^\times$ be the discs D_ϵ with the origin deleted. Then $F^{-1}(D_\epsilon^\times)$ is homeomorphic to $p_k^{-1}(D_\epsilon^\times) = D^\times_{\sqrt[k]{\epsilon}}$ which is connected. Since the a_i are accumulation points of $F^{-1}(D_\epsilon^\times) \cong D^\times_{\sqrt[k]{\epsilon}}$, $F^{-1}(D_\epsilon)$ must be connected, too, contradicting the fact that it is contained in the union of at least two nonempty open subsets with disjoint closures. Hence $n = 1$. ☐

48. Corollary [Fo, 8.4]**.** *Let $\mathcal{S} \subset Y$ be a closed discrete subset, $Y^\times := Y \backslash \mathcal{S}$. If $X^\times$ is a Riemann surface and $F^\times : X^\times \to Y^\times$ a proper unbranched holomorphic covering map $\Rightarrow$ $F^\times$ extends to a proper branched covering $F : X \to Y$ for a Riemann surface $X \supset X^\times$.*

Proof. Let $b \in \mathcal{S} \subset Y$ and consider a chart $\varphi : V_b \to \mathbb{C}$ of Y centered in b and such that $\varphi(V_b) = D$. Since $\mathcal{S}$ is a discrete subset, the domains V_b of these charts can be chosen to be disjoint. We let $V_b^\times = V_b \backslash \{b\}$. Since $F^\times$ is proper, $F^{\times -1}(V_b^\times)$ consists of a finite number of components $U_b^{i\times}$ covering $V_b^\times$ k_b^i times. Modulo biholomorphism, $F^\times|_{U_b^{i\times}}(z) = z^{k_b^i}$ by Theorem 1.46 (ii). We add ideal points a_b^i to $U_b^{i\times}$ and obtain $U_b^i := U_b^{i\times} \cup \{a_b^i\}$. We set $X = X^\times \cup \{a_b^i\}_{b \in \mathcal{S}}$. For each such a point a_b^i we define a neighbourhood basis by $\{a_b^i\} \cup (F^{\times -1}(W_b) \cap U_b^{i\times})$, where W_b runs through a neighbourhood basis of b. This turns X into a Hausdorff space, induces on $X^\times$ the given topology and defines a proper map $F : X \to Y$ in the obvious way. To define the structure of a Riemann surface on X, consider the continuation of the holomorphic maps $U_b^{i\times} \to V$ obtained by sending a_b^i to b. This gives a biholomorphic mapping $\Phi : U_b^i \to D$ corresponding to $z \in U_b^i \mapsto z^{k_b^i} \in V_b$. Since for any other chart U_1 of $X^\times$, $a_b^i \notin U_1 \cap U_b^i$, these added maps are clearly compatible with $X^\times$. Thus we obtain a Riemann surface by glueing in the local models of Corollary 1.47. In particular, $F : X \to Y$ is a proper, holomorphic map. ☐

49. Remark. In a similar vein, suppose that $F : X \to Y$, $G : Z \to Y$ are proper holomorphic covering maps, and that $\mathcal{S} \subset Y$ is a discrete subset. Then any biholomorphic map $H^\times : F^{-1}(Y \backslash \mathcal{S}) \to G^{-1}(Y \backslash \mathcal{S})$ commuting with F and G can be extended to a commuting holomorphic map $F : X \to Z$ [Fo, Theorem 8.5]. In particular, every Deck transformation $F : X^\times \to X^\times$ can be extended to a (uniquely determined) biholomorphic map $F : X \to X$ which commutes the covering map. It follows that the holomorphic structure on X in the previous Corollary 1.48 is uniquely determined.

50. Corollary and Definition (Deck transformations for branched holomorphic covering maps). We let

$$\mathrm{Deck}(X/Y) = \{G : X \to X \mid G \text{ biholomorphic }, F \circ G = F\} = \mathrm{Deck}(X^\times, Y^\times)$$

be the group of **Deck transformations of** $F : X \to Y$, where $Y^\times = Y \backslash \{\text{branch points}\}$ and $X^\times = X \backslash \{\text{ramification points}\}$.

1.3. **Analytic continuation and algebraic functions.** Our next task is to actually construct Riemann surfaces, namely as maximal domains of holomorphic functions via *analytic continuation*. Historically, this was the first instance of a Riemann surface which was not a domain in $\mathbb{C}$. This will also open a more algebraic (geometric) way of investigating Riemann surfaces through their function fields.

Analytic continuation. Let $f : U \to \mathbb{C}$ be a holomorphic function without zeroes. Locally, the holomorphic logarithm exists. What is its maximal domain of definition? The Figure 1.4 illustrates the problem for the entire holomorphic function $f(z) = z$.

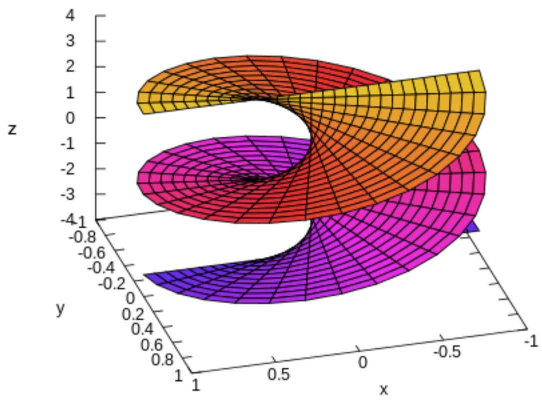

FIGURE 4. Maximal domain of the holomorphic logarithm

It is classical that the holomorphic logarithm exists on any slitted plane (cf. also Example A.8). However, if we wish to have a *maximal* domain (in a sense to be specified below) on should rather consider the covering map $\hat{X} \to \mathbb{C}$ as in Figure 1.4. By Proposition 1.40 $\hat{X}$ can be turned into a Riemann surface such that the projection onto the punctured plane $\mathbb{C}^\times$ becomes holomorphic. We will say that $\hat{X}$ was obtained by *analytic continuation* from z. Moreover, the holomorphic logarithm is globally defined, and we really obtained a pair $(\hat{X}, \log z)$ of a Riemann surface and a globally defined holomorphic function.

In the sequel, X will denote again a Riemann surface.

51. Definition (germ and stalk of holomorphic functions). Let $a \in X$. For two functions $f \in \mathcal{O}(U)$, $g \in \mathcal{O}(V)$ with $a \in U \cap V$ we say that f is equivalent to $g \Leftrightarrow$ there exists an open set $W \subset U \cap V$ with $a \in W$ and such that $f|_W = g|_W$. This is an equivalence relation whose equivalence class will be denoted by $[U, f]$ and which will be called the **germ of** f **at** a. Since the precise U is immaterial we also denote this germ by $\mathbf{f}_a$. The operations $[U, f] + [V, g] = [U \cap V, f + g]$ and $[U, f] \cdot [V, g] = [U \cap V, f \cdot g]$ turn the set

$$\mathcal{O}_{X,a} = \{[U, f] \mid a \in U,\ f \in \mathcal{O}(U)\}$$

into a $\mathbb{C}$-algebra which we call the **stalk of holomorphic functions at** a. If the underlying Riemann surface is clear from the context we simply write $\mathcal{O}_a$ for $\mathcal{O}_{X,a}$. Further, we let

$$|\mathcal{O}| = \bigsqcup_{a\in X} \mathcal{O}_a$$

be the disjoint union of all stalks and define the projection map $\pi : |\mathcal{O}| \to X$ to be the map which assigns to each $\mathbf{f}_a \in \mathcal{O}_a$ its base point $a \in X$.

52. Remark.

(i) Note that $[U, f] = \mathbf{0}_a$ = the neutral element of addition in $\mathcal{O}_{X,a} \Leftrightarrow f \equiv 0$ on some open neighbourhood of a.

(ii) Similarly, we can define the *stalk of meromorphic functions* $\mathcal{M}_{X,a}$. This also inherits the algebraic structure of $\mathcal{M}_X(U)$ and is thus a field (in fact, though we have not proven this fact yet, $\mathcal{M}_{X,a} = \operatorname{Quot} \mathcal{O}_{X,a}$ – convince yourself that $\mathcal{O}_{X,a}$ is indeed integral!).

(iii) The germ of a holomorphic function $f \in \mathcal{O}(U)$ determines f completely if U is connected, for if $[U, f] = [U, g]$, then f and g agree on some nonempty open subset of U. Hence we obtain an inclusion $\mathcal{O}(U) \hookrightarrow \mathcal{O}_a$.

The order function which we discuss next is a good example for how one uses germs:

53. The order function. For any $a \in U \subset X$ we define the **order function**

$$o_a : \mathcal{M}^*_{X,a} := \mathcal{M}_{X,a}\backslash\{\mathbf{0}_a\} \to \mathbb{Z}, \quad o_a(f) = \begin{cases} -m, & f \text{ has a pôle at } a \text{ of order } m \\ n, & f \text{ has a zero at } a \text{ of order } n > 0 \\ 0, & \text{else} \end{cases}$$

In particular, f is holomorphic on $U \Leftrightarrow o_a(f) \geqslant 0$ for all $p \in U$ and we have the

(i) *product rule:* $o_a(f \cdot g) = o_a(f) + o_a(g)$, that is we have a group morphism $(\mathcal{O}_X(U), \cdot) \to (\mathbb{Z}, +)$;

(ii) *non-archimedean property:* $o_a(f + g) \geqslant \min\{o_a(f), o_a(g)\}$.

For instance, we have $o_\infty(P) = -n$ for the meromorphic function P of the previous Example 1.31 (ii). For convenience, we will set $o_a(\mathbf{0}_a) = \infty$ if we wish to extend o_a over all of $\mathcal{M}_{X,a}$.

Of course, we could have defined the order function at a for meromorphic functions in, say, $\mathcal{M}_X(U)$, but this would be unnatural for we have to choose U, while the order only depends on the germ.

54. Remark. The order function $o_a : \mathcal{M}^*_{X,a} \to \mathbb{Z}$ is an example of a *discrete valuation.* Note that $\mathcal{O}^*_{X,a}$ is just the subring of $\mathcal{M}^*_{X,a}$ given by $o_a \geqslant 0$. It is thus a discrete valuation ring and as such in particular *local* with maximal ideal $\mathfrak{m} = \{\mathbf{f}_a \mid \o_a(\mathbf{f}_a) > 0$, that is, those germs which are no invertible near a for $\mathbf{f}_a$ has a zero.(see [AtMa, Chpater 1, 5 and 9] for a definition and further discussion of these concepts).

We topologise $|\mathcal{O}|$ as follows: For any open subset $U \subset X$ and $f \in \mathcal{O}(U)$, we let $\mathcal{W}_{U,f}$ be the open set

$$\mathcal{W}_{U,f} := \{\mathbf{f}_a \mid a \in U\} \subset |\mathcal{O}|,$$

that is, the open set $\mathcal{W}_{U,f}$ can be identified with the image of the *section* $\mathbf{f} : U \to |\mathcal{O}|$, $\mathbf{f}(a) = \mathbf{f}_a$ induced by f (recall that in general, a section of a map $\pi : X \to Y$ is a map $\sigma : Y \to X$ which satisfies $\pi \circ \sigma = \mathrm{Id}_Y$). Now a subset $\mathfrak{B} \subset \mathfrak{P}(X)$ of the power set of some set X is a basis for the topology $\Leftrightarrow$ (i) the elements $U \in \mathfrak{B}$ cover X, and (ii) for any U, $V \in \mathfrak{B}$, and $a \in U \cap V$ there exists $W \in \mathfrak{B}$ such that

$a \in W \subset U \cap V$. A basis generates a natural topology by taking the intersection of all topologies on X which contain $\mathfrak{B}$, that is, a set is open in X if and only if it is the union of sets in $\mathfrak{B}$.

55. Lemma [Fo, 6.8]. *The system $\{\mathcal{W}_{U,f}\}$ with $U \subset X$ open and $f \in \mathcal{O}(U)$ forms the basis of a topology. Furthermore, the projection becomes a local homeomorphism.*

Proof. The first assertion is straightforward, see for instance [Fo, Theorem 6.8]. To show that the projection is a local homeomorphism, suppose that $\mathbf{f}_a = [U, f] \in |\mathcal{O}|$ and $\pi(\mathbf{f}_a) = a$. Then $\mathbf{f}_a \in \mathcal{W}_{U,f}$ is an open neighbourhood of $\mathbf{f}_a$, and U is an open neighbourhood of $a \in X$. The projection restricted to $\mathcal{W}_{U,f}$ in injective and thus a homeomorphism onto its image U. □

56. Remark. The Identity Theorem for holomorphic functions implies that $|\mathcal{O}|$ is Hausdorff, see for instance [Fo, Theorem 6.10]. It follows by Proposition 1.40 that any connected component of $|\mathcal{O}|$ is a Riemann surface. In particular, any function germ $\mathbf{f}_a \in |\mathcal{O}|$ singles out a Riemann asurface $\hat{X}$. Further, it defines a holomorphic function $\hat{f} \in \mathcal{O}(\hat{X})$ such that $\hat{f}(\mathbf{f}_a) = f(a)$. Indeed, let $\hat{f}(\mathbf{g}_b) = g(b)$ for any $\mathbf{g}_b \in \hat{X}$. In particular, $\hat{f}|_{\mathcal{W}_{V,g}} = g \circ \pi$. Note in passing that $\hat{X}$ cannot be compact for $\hat{f}$ is not constant.

57. Definition. Let $u : I = [0, 1] \to X$ be a curve from a to b. The germ $\psi \in \mathcal{O}_b$ is said to be obtained by **analytic continuation along the curve u from the germ** φ if there exists a family $\varphi_t \in \mathcal{O}_{u(t)}$, $t \in I$ with $\varphi_0 = \varphi$ and $\varphi_1 = \psi$ and such that φ_t is locally induced by a holomorphic function, i.e. for all $\tau \in I$ there exists a neighbourhood I_τ and an open subset U_τ containing $u(I_\tau) \subset X$ with $f \in \mathcal{O}(U_\tau)$ such that $\mathbf{f}_{u(t)} = \varphi_t$.

58. Remark. Since I is compact every open cover of $u(I)$ can be reduced to a finite cover. Therefore the condition of the previous definition can be reformulated as follows: *There exists a partition $0 = t_0 < t_1 < \ldots < t_{n-1} < t_n = 1$ of I as well as open connected sets $U_i \subset X$ with $u[t_{i-1}, t_i] \subset U_i$ and $f_i \in \mathcal{O}(U_i)$ such that*

(i) $\varphi = \mathbf{f}_{1,u(0)}$, $\psi = \mathbf{f}_{n,u(1)}$;
(ii) $f_i|_{V_i} = f_{i+1}|_{V_i}$, *where V_i denotes the connected component of the intersection $U_i \cap U_{i+1}$ containing the point $u(t_i)$,*

see also Figure 1.5

Note that by definition, $\varphi = \mathbf{f}_a$ for some holomorphic function defined near a; Definition 1.57 requires this choice to be uniform near φ, i.e. the same function does the job for any φ_1 sufficiently close to φ. In this way this condition can be seen as a continuity property of φ_t. Indeed, we have the

59. Lemma [Fo, 7.2]. *A function germ $\psi \in \mathcal{O}_b$ is the analytic continuation of $\varphi \in \mathcal{O}_a$ along $u : I \to X \Leftrightarrow$ there exists a lifting $\hat{u} : I \to |\mathcal{O}|$ of u such that $\hat{u}(0) = \varphi$ and $\hat{u}(1) = \psi$.*

In particular, the analytic continuation of a germ, if it exists, is uniquely determined

Proof. $\Rightarrow$) By design, $\varphi_t = \hat{u}$ is the desired lifting (this is precisely the reason why we required φ_t to be locally induced by a holomorphic function).

$\Leftarrow$) Let $\varphi_t = \hat{u}(t)$ and $\tau \in I$. There exists a neighbourhood $\mathcal{W}_{U,f}$ of $\hat{u}(\tau) \in |\mathcal{O}|$ such that $\varphi_t = \hat{u}(t) = \mathbf{f}_{u(t)}$ for all $t \in u^{-1}(U) = I_\tau$. Hence φ_t is the analytic continuation of φ to ψ. □

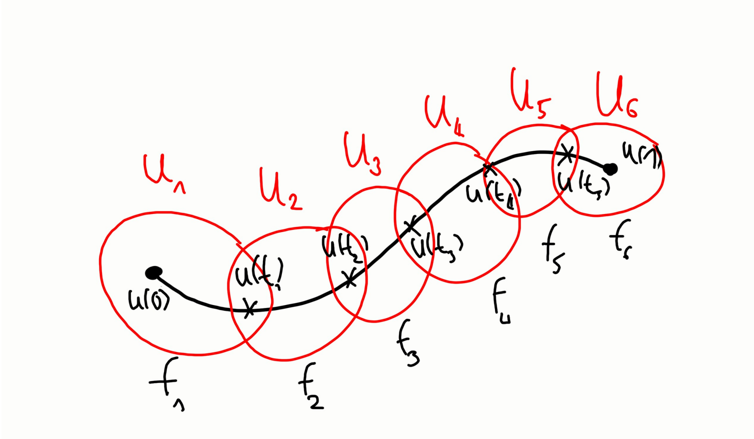

FIGURE 5. Analytic continuation along u.

From Proposition B.13 we immediately deduce the first part of the

60. Monodromy Theorem [Fo, 7.3]. *Let $\varphi \in \mathcal{O}_a$ and $u_{0,1} : I \to X$ be homotopic curves from a to b via the homotopy u_s, $s \in I$. Assume that for every u_s, there exists an analytic continuation of $\varphi \in \mathcal{O}_a$ to $\hat{u}_s(1) \in \mathcal{O}_b$. Then the endpoint does not depend on s, i.e. $\hat{u}_0(1) = \hat{u}_s(1) = \hat{u}_1(1)$ for all s.*

Therefore, if X is simply connected and $\varphi \in \mathcal{O}_a$ is a germ which can be analytically continued along every curve starting at $a \Rightarrow$ there exists a globally defined holomorphic function $f \in \mathcal{O}(X)$ such that $\mathbf{f}_a = \varphi$.

Proof. Only the last assertion requires justification. Let $\varphi_b = [U, g] \in \mathcal{O}_b$ be the germ obtained by analytic continuation along some path from $\varphi_a \in \mathcal{O}_a$. Since X is simply-connected, this does not depend on the path for any closed loop can be retracted to a point, that is, any two paths with the same inital and final points are homotopic. Then $f(b) := g(b)$ yields the desired holomorphic map. $\square$

61. Example. Consider the function $f(z) = z$. Then f has no zeroes on $\mathbb{C}^*$ so we can locally take its logarithm. Let $u(t) = e^{2\pi i t} \subset \mathbb{C}^*$ be the unit circle and let $t_j = 2\pi j/3$ for $j = 0, \ldots, 3$. On a suitable open neighbourhood of $u([t_{j-1}, t_j])$, $j = 1$, 2 or 3, we define $g_j(z) = \log f = \log|z| + i\arg_j z$, where $\arg_j$ takes values in $[2\pi(j-1)/3 - \epsilon, 2\pi j/3 + \epsilon]$ for some small $\epsilon > 0$, see Figure 1.6. Then g_j agree on the overlaps and thus define an analytic continuation of the germ $\mathbf{g}_{1,u(0)}$. However, the circle lies in a not simply connected domain, and indeed, $g_3(u(1)) = g_3(u(0)) = 2\pi i \neq g_1(u(1)) = 0$.

On the other hand, as follows from Remark 1.56, any germ $\mathbf{g}_a$ of a locally defined holomorphic logarithm $g = \log z$ singles out a Riemann surface $\hat{X}$ inside $\pi : |\mathcal{O}| \to \mathbb{C}^*$. The corresponding function $\hat{g} \in \mathcal{O}(\hat{X})$ extends g to $\pi(\hat{U})$ where $\hat{U}$ is a maximal open neighbourhood of $\mathbf{g}_a$ such that $\pi|_{\hat{U}}$ is injective. To generalise this observation, let $\pi : \hat{X} \to X$ be an unbranched holomorphic map. Then π induces an isomorphism $\pi_b^* : \mathcal{O}_{X,\pi(b)} \to \mathcal{O}_{\hat{X},b}$ for any $b \in \hat{X}$ by setting $\pi_b^*[U, f] = [\pi^{-1}(U), f \circ \pi]$ since π

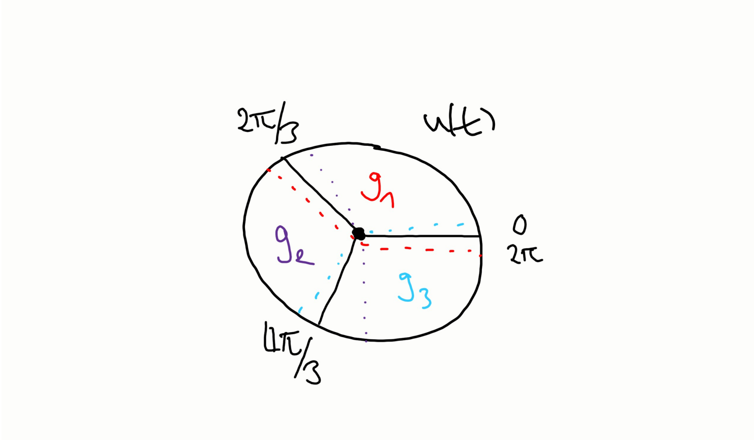

FIGURE 6. Analytic continuation of $\log z$ along the unit circle.

is locally biholomorphic. We let $\pi_{b*} = (\pi_b^*)^{-1} : \mathcal{O}_{\hat{X},b} \to \mathcal{O}_{X,\pi(b)}$ be its inverse (we sometimes drop the base point b to ease notation).

62. Definition (analytic continuation). Let $\varphi \in \mathcal{O}_{X,a}$. A quadrupel $(\hat{X}, \pi, \hat{f}, \hat{a})$ is called an **analytic continuation of** φ if

(i) $\hat{X}$ is a Riemann surface and $\pi : \hat{X} \to X$ is an unbranched holomorphic map;
(ii) $\hat{f} \in \mathcal{O}(\hat{X})$;
(iii) $\pi(\hat{a}) = a$ and $\pi_*(\hat{\mathbf{f}}_{\hat{a}}) = \varphi$.

An analytic continuation $(\hat{X}, \pi, \hat{f}, \hat{a})$ of φ is **maximal** if the following universal property is satisfied: For any other analytic continuation (Z, q, g, b) of φ there exists a holomorphic map $F : Z \to \hat{X}$ such that $F(b) = \hat{a}$, $g = F^*\hat{f} = \hat{f} \circ F$ and $\pi \circ F = q$:

$$\begin{tikzcd} & (\hat{X}, \hat{f}, \hat{a}) \arrow[d, "\pi"] \\ (Z, g, b) \arrow[ur, dashed, "F"] \arrow[r, "q"] & (X, \varphi) \end{tikzcd}$$

As usual, universality implies that a maximal analytic continuation is essentially uniquely determined. Guided by the example of the holomorphic logarithm we prove the

63. Theorem [Fo, 7.8]**.** *For any function germ $\varphi \in \mathcal{O}_{X,a}$ there exists a maximal analytic continuation, namely $(|\mathcal{O}_X|_\varphi, \pi, \hat{f}, \varphi)$ where $|\mathcal{O}_X|_\varphi$ is the Riemann surface of $|\mathcal{O}_X|$ distinguished by φ, and $\hat{f}$ is the natural function on $|\mathcal{O}_X|_\varphi$, cf. Remark 1.56.*

This requires first a lemma.

64. Lemma [Fo, 7.7]**.** *Let $(\hat{X}, \pi, \hat{f}, \hat{a})$ be an analytic continuation of $\varphi \in \mathcal{O}_{X,\pi(\hat{a})}$. If $\hat{u} : I \to \hat{X}$ is a curve with $\hat{u}(0) = \hat{a}$, then $\varphi_t := \pi_*\hat{\mathbf{f}}_{\hat{u}(t)}$ is an analytic continuation of φ along $u = \pi \circ \hat{u}$.*

Proof. We need to show that locally, there exists $g \in \mathcal{O}_X(U)$ such that $\mathbf{g}_{u(t)} = \varphi_t$. Indeed, since π is an unbranched holomorphic map, $\pi|_{\hat{U}} : \hat{U} \subset \hat{X} \to U \subset X$ is biholomorphic for suitable open sets U and $\hat{U}$. Put $g := (\pi|_{\hat{U}})^{-1*}\hat{f} = \hat{f} \circ (\pi|_{\hat{U}})^{-1} : U \to \mathbb{C}$. By design, $\mathbf{g}_{u(t)} = \pi_* \hat{\mathbf{f}}_{\hat{u}(t)} = \varphi_t$. □

Proof. (of Theorem 1.63) Let $\hat{X}$ be the connected component of $|\mathcal{O}|$ containing φ, $\pi : \hat{X} \to X$ the restriction of $|\mathcal{O}| \to X$, and $b = \varphi \in |\mathcal{O}|$. We let $\hat{f} : \hat{X} \to \mathbb{C}$ be the corresponding holomorphic function, i.e. $\hat{f}(\mathbf{f}_x) = f(x)$. It follows that f is holomorphic, and $f(b) = a$. Thus $(\hat{X}, \pi, f, b)$ is an analytic continuation of φ.

To show that it is maximal, suppose that (Z, q, g, b) is another analytic continuation of φ. We define $F : Z \to \hat{X}$ as follows. Let $z \in Z$ and choose a curve $\hat{u}$ from b to z. According to Lemma 1.64 the function germ $q_*(\mathbf{g}_z) \in \mathcal{O}_{X,q(z)} \subset |\mathcal{O}|$ is obtained via analytic continuation from φ along the curve $u = q \circ \hat{u}$. It follows that $q_*(\mathbf{g}_z)$ lies precisely in the connected component determined by φ which is just $\hat{X}$. Hence the map $F : Z \to \hat{X}$, $z \mapsto q_*(\mathbf{g}_z)$ is well-defined. It is easy to see that F satisfies all the required properties. □

Field extensions and algebraic functions. Recall from algebra that a *function field of n variables* is a finite extension of a field of the form $k(x_1, \ldots, x_n)$ for algebraically independent variables x_i. For instance, $K_{\mathbb{P}^1} = \mathbb{C}(t)$ is a function field of one variable. While this justifies the term "function field" for $\mathbb{P}^1$ we now wish to show that K_X is a function field of one variable for any compact Riemann surface X. This will be done in two steps. First, consider a branched n-sheeted holomorphic covering map $\pi : X \to Y$. Pull-back by π induces a morphism of fields $\pi^* : K_Y \to K_X$ which sends f to $\pi^* f = f \circ \pi$. Since this map is nontrivial it is necessarily injective, that is, we can regard K_Y as a subfield of K_X. Furthermore, we are going to show that $[K_X : K_Y] = n$ so that π^* defines a *finite* (and in particular, algebraic) field extension. In a second step (to be carried out later) we show that every compact Riemann surface admits a branched holomorphic covering $X \to \mathbb{P}^1$ from which it follows that K_X is a function field of one variable.

Our first goal os to show that $[K_X : K_Y] \leqslant n$ if $\pi : X \to Y$ is a branched n-sheeted holomorphic covering. We first need to introduce some technicalities. Let $\pi : X \to Y$ be an n-sheeted unbranched holomorphic covering map, and let $f \in K_X$. For a special neighbourhood V we therefore have $f^{-1}(V) = \bigcup_{j=1}^k U_j$ with $\pi|_{U_j} : U_j \to V$ invertible. Let $f_j = f \circ (\pi|_{U_j})^{-1} \in \mathcal{M}_Y(V)$. We consider the polynomial

$$P_{V,f}(T) = \Pi_{j=1}^k (T - f_j) = \sum_{j=0}^n \sigma_j T^j \in \mathcal{M}_Y(V)[T], \tag{4}$$

where $\sigma_j = \sigma(f) = (-1)^j s_{n-j}(f_1, \ldots, f_n)$ are given by the $n-j$-th symmetric polynomial s_{n-j} (e.g. $s_0(x_1, \ldots, x_n) = 1$, $s_1(x_1, \ldots, x_n) = \sum x_j, \ldots, \sigma_n(x_1, \ldots, x_n) = \Pi x_j$). If we consider another special neighbourhood W with $f^{-1}(W) = \bigcup_{j=1}^k \tilde{U}_j$, then the functions $\tilde{f}_j = f \circ (\pi|_{\tilde{U}_j})^{-1}$ agree with f_j on the intersection $V \cap W$ up to some relabeling, that is, $P_{V,f}(T)|_{V \cap W} = P_{W,f}(T)|_{V \cap W}$ for $\sigma_j(f_1, \ldots, f_n) = \sigma_j(\tilde{f}_1, \ldots, \tilde{f}_n)$. Indeed, the σ_j are symmetric in their arguments, i.e. invariant under the action of the permutation group $\mathfrak{S}_n$, and thus invariant under relabeling.

Hence the locally defined σ_j piece together to give globally well-defined meromorphic functions $\sigma_j(f) \in K_Y$, $j = 0, \ldots, n$ called the **elementary symmetric functions of** f. Summarising, we obtain a polynomial $P_f(T) \in K_Y[T]$ which satisfies $P_f(f(a))(\pi(a)) = 0$ for all $a \in X$, that is, $\pi^* P_f \in K_X[T]$ satisfies

$$\pi^* P_f(f) = 0.$$

More generally, the elementary symmetric functions are well-defined for branched covering maps $\pi : X \to Y$. Indeed, let $\mathcal{C} \subset Y$ be a closed discrete subset of Y which contains all critical values of π. Further, let $\mathcal{S} = \pi^{-1}(\mathcal{C})$, $X^\times := X \backslash \mathcal{S}$ and $Y^\times := Y \backslash \mathcal{C}$ so that in particular, $\pi|_{X^\times} : X^\times \to Y^\times$ becomes an unbranched holomorphic covering. For $f \in \mathcal{M}_X(X^\times)$ consider the elementary functions $\sigma_j(f) \in \mathcal{M}_Y(Y^\times)$. In the following, we say that a meromorphic function f extends meromorphically to a if $z^m f$ is bounded near a for a local coordinate z with $z(a) = 0$.

65. Lemma [Fo, 8.2]**.** *f can be continued holomorphically resp. meromorphically for all $a \in \mathcal{S}$ $\Leftrightarrow$ the $\sigma_j(f)$, $j = 1, \ldots, k$ can be continued holomorphically resp. meromorphically to $b = \pi(a) \in Y$. In particular, the elementary functions of $f \in \mathcal{M}_X(X)$ are also defined in case of a branched holomorphic covering.*

Proof. Assume first that f can be continued holomorphically to all $a \in \pi^{-1}(b)$, $b \in \mathcal{C}$. Then $\sigma_j(f)$ exists outside b and is bounded, hence $\sigma_j(f)$ can be extended to b by Riemann's Removable Singularities Theorem. Conversely, substituting $T = f$ into $\pi^* P_f(T)$ and evaluating at $x \in X^\times$ we get

$$f^n(x) + \sigma_1(f)(\pi(x)) f^{n-1}(x) + \ldots + \sigma_n(f)(\pi(x)) = 0$$

so that σ_j bounded near b implies f bounded near $a \in \pi^{-1}(b)$ so that f extends.

Secondly, assume that f extends meromorphically to X. Thus, if z is a local coordinate for Y with $z(b) = 0$, $b \in \mathcal{C}$, then $w = \pi^* z$ is a local coordinate around $a \in \pi^{-1}(b)$, and $w^m f$ extends holomorphically over a for m big enough. In particular, $\sigma_j(w^m f) = z^{mj} \sigma_j(f)$ is holomorphic by the first case, hence $\sigma_j(f)$ extends meromorphically to b. Conversely, if $z^m \sigma_j(f)$ can be continued holomorphically to b for all j, then $w^m f$ can be continued holomorphically to all $a \in \pi^{-1}(b)$, hence f extends meromorphically. ☐

66. Remark. The proof also applies if X is a disconnected union of Riemann surfaces, e.g. the trivial cover of Y by n copies of Y.

From the identity $(\pi^* P_f)(f) = f^k + \sum_{j=1}^k (\pi^* \sigma_j(f)) f^{n-j} = 0$ we directly deduce the

67. Theorem [Fo, 8.3]**.** *Let $\pi : X \to Y$ be a branched holomorphic n-sheeted covering map. If $f \in K_X$ with elementary symmetric functions $c_1, \ldots, c_n$ $\Rightarrow$*

$$f^n + (\pi^* \sigma_1(f)) f^{n-1} + \ldots + \pi^* \sigma_{n-1}(f) f + \pi^* \sigma_n(f) = 0.$$

In particular, considering K_Y as a subfield of K_X via π^* this defines a polynomial relation on $f \in K_X$ with coefficients in K_Y so that $K_Y \subset K_X$ is a finite field extension of degree $\leqslant n$.

68. Remark. We will see later that the degree is actually equal to n.

Summarising, we have seen that any holomorphic branched covering $\pi : X \to Y$ gives rise to a finite field extension $K_Y \subset K_X$. Conversely we can ask when a finite field extension of K_Y can be realised by a branched holomorphic covering map? Finite field extensions of k arise by adjoining roots of irreducible polynomials $P \in$

$k[T]$. In fact, since we are working with fields of characteristic zero, the existence of a primitive element (cf. Theorem D.10) says that any finite field extension $K_Y \subset L$ is of the form $K_Y(f) \cong K_Y[T]/(P)$ where $P \in K_Y[T]$ is an irreducible polynomial. We wish to find a Riemann surface X with $K_X = L$.

69. Theorem [Fo, 8.7-9]**.** *Let $P \in K_Y[T]$ be an irreducible polynomial of degree n $\Rightarrow$ there exists a Riemann surface X and a branched n-sheeted holomorphic covering $\pi : X \to Y$ such that P has a root in K_X, that is, there exists $f \in K_X$ such that $\pi^*P(f) = 0$. The triple (X, π, f) is unique up to biholomorphic mappings commuting with the covering maps and pulling one meromorphic function back to the other.*

Proof. We proceed in three steps.

Step 1. *If $P(T) = T^n + \sum_{j=1}^n c_j T^j \in \mathcal{O}_{X,a}[T]$ and $\sum_{j=0}^n c_j(a)T^{n-1} \in \mathbb{C}[T]$ has simple zeroes $z_1, \ldots, z_n \Rightarrow$ there exist germs $\varphi_1, \ldots, \varphi_n \in \mathcal{O}_{X,a}$ such that $\varphi_j(a) = z_j$ and $P(T) = \Pi_{j=1}^n (T - \varphi_j)$.* Indeed, let $c_1, \ldots, c_n$ be holomorphic functions on the disk $D_R \subset \mathbb{C}$. Consider the holomorphic function $f(w, z) = w^n + \sum_{j=1}^n c_j(z)w^{n-j}$. If w_0 is a simple zero of the polynomial $f(\cdot, 0)$, then it follows from the Implcit Function Theorem (which holds also for holomorphic functions) that there exists a holomorphic function $\varphi : D_r \to \mathbb{C}$, $0 < r \leqslant R$ with $\varphi(0) = w_0$ and $f(\varphi(z), z) = 0$ (see also [Fo, Lemma 8.7] for a proof avoiding the use of the IFT).

Step 2. *Existence.* Let $\Delta = \Delta(y) \in K_Y$ be the discriminant of P. This is a certain polynomial in the coefficients of P with $\Delta(y) = 0 \Leftrightarrow P[T](y) = P'[T](y) = 0$ (where P' is the formal derivative of P in T). Since P is irreducible, Δ does not vanish identically. It follows that there exists a discrete subset $\mathcal{S} \subset Y$ such that for all $y \in Y^\times := Y \backslash \mathcal{S}$, $\Delta(y) \neq 0$ and $c_j \in \mathcal{O}_Y(Y^\times)$ are holomorphic. Let $X^\times \subset |\mathcal{O}| \to Y$ be the set of all germs $\varphi \in \mathcal{O}_{Y,y}$, $y \in Y^\times$ such that $P(\varphi) = 0$. Let $\pi^\times : X^\times \to Y^\times$ be the restriction of the natural projection $|\mathcal{O}| \to Y$. For any $y \in Y^\times$, the polynomial

$$p_y(T) := T^n + \sum_{j=1}^n c_j(y)T^{n-j} \in \mathbb{C}[T]$$

has precisely n distinct zeros for $\Delta(y) \neq 0$. By the previous step it follows that for every $y \in Y'$ there exists an open neighbourhood V of y and functions $f_i \in \mathcal{O}_Y(V)$ such that $P(T) = \Pi_{j=1}^n (T - f_j)$ on V. Further, $\pi^{\times -1}(V) = \bigcup_{j=1}^n U_j$ where $U_j = \mathbf{f_j}(V)$ is the image of the section of $|\mathcal{O}|$ induced by f_j is a disjoint union of open sets (the zeros of p_y are simple!). Further, $\pi^\times|_{U_j} \to V$ is a homeomorphism which shows that $\pi^\times : X^\times \to Y^\times$ is a covering map. We claim that $X^\times$ is connected so that $\pi^\times$ extends to a branched holomorphic cover $\pi : X \to Y$ by Proposition 1.48. If not, assume for simplicity that there are two connected components $X_{1,2}$. We can regroup the product $P(T) = \Pi_{j=1}^n (T - f_j) = \Pi_{j=1}^{n_1} (T - f_{k_j}) \cdot \Pi_{j=1}^{n_2} (T - f_{i_j})$ where the two factors comprise the functions giving rise to germs in X_1 and X_2 respectively. By Lemma 1.65 these piece together to two meromorphic functions $P_{1,2}(T) \in K_Y[T]$. It follows that $P(T) = P_1(T)P_2(T)$, contradicting the irreducibility of P. Finally, let f be the tautological holomorphic function on X. Then $\pi^*P(f) = f^n(x) + \sum_{j=1}^n c_j(\pi(x))f^j(x) = 0$. By Lemma 1.65 again, we can extend f to a meromorphic function on all of X such that $\pi^*P(f) = 0$.

Step 3. *Uniqueness.* We briefly sketch uniqueness, for details see [Fo, Theorem 8.9]. Suppose (Z, q, g) is another triple with the required properties. Let $\mathcal{T} \subset Z$ be the union of the poles of g and the branch points of q. Let $\mathcal{T}' = q(\mathcal{T}) \subset Y$ and $X' = \pi^{-1}(Y^\times \backslash \mathcal{T}') \subset X^\times$. To construct a map $F : Z \backslash \mathcal{T} \to X'$, take $z \in Z$ and let $y = q(z)$. Then $q_*\mathbf{g}_z \in \mathcal{O}_{Y,y}$ solves $P(q_*\mathbf{g}_z) = 0$ and thus must be in X' by design of $X^\times$. It follows that $F(z) = q_*\mathbf{g}_z$ is a continuous map which commutes with q and π. By Remark 1.49 this extends to a holomorphic map $Z \to X$ which commutes with π and q. It is easy to check that F is in fact biholomorphic.

□

Since f is the root of the polynomial equation $P = 0$ it is called the **algebraic function defined by** P. We also say that (X, π, f) is the algebraic function determined by (Y, P).

70. Example. We have seen in Example 1.34 that $K_{\mathbb{P}^1} \cong \mathbb{C}(Z)$, the rational functions in one variable. Hence for any polynomial $P(T)$ with coefficients in the ring $\mathbb{C}(Z)$ there exists a finite branched covering $\pi : X \to \mathbb{P}^1$ with X a (compact) Riemann surface and $f \in K_X$ such that $\pi^*P(f) = 0$. If, for instance, we consider $P(T) = T^2 - g(Z) \in K_{\mathbb{P}^1}[T]$ for some polynomial $g(Z) \in \mathbb{C}[Z]$, then this defines a compact Riemann surface X on which the (meromorphic) function $\sqrt{g}$ is defined.

71. Corollary [Fo, 8.12]**.** *If (X, π, f) is an algebraic function defined by (Y, P) with $\deg P = n$, then $K_X = K_Y(f) \cong K_Y[T]/(P)$. In particular, $K_Y \subset K_X$ is an algebraic field extension of degree n.*

Proof. By Theorem 1.67 the field extension $K_Y \subset K_X$ is algebraic. Let $\mu \in K_Y[T]$ be the minimal polynomial of some $f_0 \in K_X$ such that its degree d is maximal. Since P is irreducible, $d \geqslant n$. We claim that $K_X = K_Y(f_0)$. Indeed, let $g \in K_X$. Since K_Y is a perfect field (it has characteristic zero), we can find a primitive element for the field extension $K_Y \subset K_Y(f_0, g) \subset K_X$, that is, $K_Y(f_0, g) = K_Y(h)$ for some $h \in K_Y(f_0, g)$, cf. Theorem D.10. By definition of d, $[K_Y(h) : K_Y] = \dim_{K_Y} K_Y(h) \leqslant d$, but $d \geqslant \dim_{K_Y} K_Y(f_0, g) \geqslant \dim_{K_Y} K_Y(f_0) = d$, whence equality of dimension. Since $K_Y(f_0)$ is a subspace of $K_Y(f_0, g)$ we have $K_Y(f_0) = K_Y(f_0, g)$ and thus $K_Y(f_0) = K_X$. By Theorem 1.69, $n \geqslant \dim_{K_Y} K_X = d \geqslant \dim_{K_Y} K_Y(f)$. On the other hand , $\dim_{K_Y} K_Y(f) = n$ for P is irreducible of degree n. Hence $d = n$ and $K_X = K_Y(f) \cong K_Y[T]/(P)$. □

72. Explicit construction of an algebraic function. It is instructive to consider a special case of Theorem 1.69. Consider a polynomial $g(z) = \Pi_{j=1}^n(z - a_j)$ with n distinct roots $a_1, \ldots, a_n$ which we consider as a meromorphic function on $\mathbb{P}^1$. $P(T) = T^2 - f$ is irreducible over $K_{\mathbb{P}^1} = \mathbb{C}(z)$ (it has no zeroes), so it defines an algebraic function suggestively denoted $(X, \pi, \sqrt{g})$. We are going to construct $\pi : X \to Y$ explicitly.

Let $\mathcal{S} = \{a_1, \ldots, a_n\} \cup \{\infty\}$ (in ∞, f has possibly a singularity), and $Y^\times = \mathbb{P}^1 \backslash \mathcal{S}$ and $X^\times = \pi^{-1}(Y^\times) \subset |\mathcal{O}_Y|$ be the 2-sheeted unbranched covering given by the germs φ which solve $\varphi^2 - \mathbf{g}_y = 0$. In particular, we can analytically continue any germ in $\mathcal{O}_{Y^\times, y}$ along any curves in $Y^\times$. Now consider what is happing near $\mathcal{S}$. For $j \in \{1, \ldots, n\}$ we choose sufficiently small discs $D_j := D_\epsilon(a_j)$ such that $D_j \cap \mathcal{S} = \{a_j\}$. The functions $g_j(z) = \Pi_{k \neq j}(z - a_k)$ have no zeroes in D_j so that there exist holomorphic functions $h_j : D_j \to \mathbb{C}$ with $h_j^2 = g_j$. In particular, $g(z) = (z - a_j)h_j^2$ on D_j. Consider the curve $u(t) = a_j + re^{it} \subset D_j$. For a given t_0 we get the germ $\varphi_{t_0} = \sqrt{r}e^{it/2}\mathbf{h}_{u(t_0)}$. If we continue this germ around the circle we get $-\varphi_{t_0}$ after completing one loop. This means that $\pi : \pi^{-1}(D_j^\times) \to D_j^\times = D_j \backslash \{a_j\}$ is the connected $2-1$ covering isomorphic to $z \mapsto z^2$, cf. Theorem 1.46 (otherwise it would be disconnected and one would find again φ_{t_0} after the completion of one circle). So we add a single point to $X^\times$ over a_j. Locally, π looks like $z \mapsto z^2$. Now this constructs the covering over $\mathbb{C} \cong \mathbb{P}^1\backslash\{\infty\}$. Next we have a look at $D_\infty := D_\epsilon(\infty)$, a disc which is also taken to be so small that $D_\infty \cap \mathcal{S} = \{\infty\}$. There, we can write $f(w) = w^n f_0(w)$ for a local coordinate w near ∞ and holomorphic $f_0(w) \neq 0$ on D_∞. If n is odd we can write $f(w) = wh^2(w)$ for h meromorphic, and $f(w) = h(w)$ if n is even. As before we can consider a circle path around ∞. In the odd case, continuation of the germ going around the loop yields minus the germ and we have to add one point over ∞, while the covering $\pi : X \to Y$ is disconnected near ∞ if n is even, that is, we have to add two points.

Field extensions can be also analysed using *Galois theory*, see Appendix D. Let $\pi : X \to Y$ be a branched holomorphic covering with group of Deck transformations $\mathrm{Deck}(X/Y)$ (see Definition 1.50). We explore the relation between the Galois group $\mathrm{Gal}(K_X/K_Y)$ and the group of Deck transformations next.

First, we define a representation $\mathrm{Deck}(X/Y) \to \mathrm{Gal}(K_X/K_Y)$ as follows. If $F \in \mathrm{Deck}(X/Y)$ and $f \in K_X$, define

$$\sigma_F(f) := F^{-1*} \circ f = f \circ F^{-1}. \tag{5}$$

Taking the inverse actually ensures that we have representation, for

$$(\sigma_F \circ \sigma_G)(f) = f \circ (G^{-1} \circ F^{-1}) = f \circ (F \circ G)^{-1} = \sigma_{F\circ G}(f).$$

Clearly, σ_F is the identity on K_Y.

73. Definition (Galois covering). Let $\pi : X \to Y$ be a branched holomorphic covering map. Let $\mathcal{C} \subset Y$ be the set of critical values of π. The covering is called **Galois** if the unbranched cover $X' = X\backslash\pi^{-1}(\mathcal{C}) \to Y' = Y\backslash\mathcal{C}$ is Galois, i.e. $\mathrm{Deck}(X'/Y')$ acts transitively on each fibre (cf. Definition B.27).

74. Theorem (Galois correspondence). *Let (X, π, f) be the algebraic function determined by (Y, P) $\Rightarrow$ the representation* (5) *induces an isomorphism*

$$\mathrm{Deck}(X/Y) \cong \mathrm{Gal}(K_X/K_Y).$$

Moreover, $\pi : X \to Y$ is Galois if and only if the field extension $K_Y \subset K_X$ is Galois.

Proof. By construction, $\sigma_F(f) \neq f$ for any nontrivial Deck transformation $F \in \mathrm{Deck}(X/Y)$ which shows that the reprensetation is faithful, i.e. injective. To show surjectivity, let $\sigma \in \mathrm{Gal}(K_X/K_Y)$. Then $(X, \pi, \sigma(f))$ is also an algebraic function so that there exists a Deck transformation $F : X \to X$ with $f \circ F = \sigma(f)$. Hence $\sigma = \sigma_{F^{-1}}$, for $K_X = K_Y(f)$ so that σ is determined by th evalue it takes on f. Finally, $\pi : X \to Y$ resp. $K_Y \subset K_X$ is Galois if and only if $\mathrm{Deck}(X/Y)$ resp. $\mathrm{Gal}(X/Y)$ contains n elements, see Definition B.27 and [Bo, 4.1.3 and 4.1.4]. $\square$

Plane algebraic curves. Finally, we sketch another way of representing *compact* Riemann surfaces – namely as *plane algebraic curves*. This will also show (modulo a result we will establish later) that the function field K_X completely determines X. This paragraph requires some basic knowledge on projective spaces.

As we have already mentioned any compact surface admits a nonconstant meromorphic function which defines a branched n-sheeted holomorphic map $\pi : X \to \mathbb{P}^1$ (as mentioned above, this is a nontrivial analytical fact; once this has been established, the theory of Riemann surfaces becomes completely algebraic). In particular, K_X is a finite extension of $K_{\mathbb{P}^1} \cong \mathbb{C}(z)$, that is, there exists $f \in K_X$ and a uniquely determined monic irreducible polynomial $P[T] \in \mathbb{C}(z)[T]$ such that $P(f) = 0$ of degree n. After clearing the denominators we can regard P as an element in $\mathbb{C}[z, T]$ which we write as $\mathbb{C}[w, z]$ to make it look more symmetric. Let $n = \deg P$. Now take its associated homogeneous form $Q(u, w, z) = u^n P(w/u, z/u)$. This is now a homogeneous polynomial for which we consider the zero locus $C = \mathcal{Z}(Q) \subset \mathbb{P}^2$, the associated **plane algebraic curve**.

75. Proposition [Gu, Lemma 31]**.** *With every plane algebraic curve we can associate a Riemann surface $X(C)$ in a natural way.*

Proof. We first restrict attention to $U = \{[u : v : w] \mid u \neq 0\} \cong \mathbb{C}^2 \subset \mathbb{P}^2$. Then C is given as $C_u = \{(v, w) \mid P(v, w) = 0\} \subset \mathbb{C}$. Consider $P(z, w)$ as a polynomial in $\mathbb{C}(w)$, and let $\Delta(w)$ be its discriminant (a polynomial in w). Generically, $\Delta(w) \neq 0$ and at such a point, the polynomial $P(v, w)$ has precisely n distinct roots $v_1, \ldots, v_n$. This entails $\partial_v P(v_i, w)$ so that by the implicit function theorem, $C_u^\times = C_u \cap (\Delta^{-1}(0))^c$ is locally given by holomorphic functions $(\varphi_i(w), w)$. In particular, projection on the second factor defines an unbranched covering map $\pi : C_u^\times \to (\Delta^{-1}(0))^c \subset \mathbb{C}$. By Corollary 1.41 this extends to a branched holomorphic covering map $C_u \to \mathbb{C}$ which gives C_u a uniquely determined holomorphic structure. Next repeat this argument on the sets $V = \{v \neq 0\}$ and $W = \{w \neq 0\}$. Since the holomorphic structure is uniquely determined on the corresponding Riemann surfaces C_v and C_w, they all agree on the overlaps and glue thus to a globally defined Riemann surface $X(C)$. □

It is true, though we cannot prove it yet, that if C is associated with X, then $X \cong X(C)$. In particular, we have the

76. Corollary [Gu, Corollary to Theorem 27]**.** *The function field K_X determines X up to biholomorphic maps.*

2. The theorem of Riemann-Roch

Having discussed the general structure of Riemann surfaces we next analyse general properties of an abstract compact Riemann surface X.

2.1. **Differential forms, sheaves and cohomology.** In order to investigate X further, we first need to introduce a higher form of live than ordinary holomorphic functions, namely *differential forms.*

Differential forms. Let $U \subset \mathbb{C}$ be an open subset. We identify $\mathbb{C}$ with $\mathbb{R}^2$ in the standard way, namely $z = x + iy$. As before we denote by $C^\infty(U)$ the $\mathbb{C}$-algebra of functions $f : U \to \mathbb{C} \cong \mathbb{R}^2$ which are smooth. Apart of the usual derivation operators ∂_x aand ∂_y we introduce

$$\partial := \partial_z := \frac{1}{2}(\partial_x - i\partial_y) \quad \text{and} \quad \bar{\partial} := \partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y).$$

As explained in the Appendix A, the kernel of $\partial : C^\infty(U) \to C^\infty(U)$ is just $\mathcal{O}(U)$, the $\mathbb{C}$-algebra of holomorphic functions.

Recall that a function $f : U \to \mathbb{C}$ for U an open set of a Riemann surface X is smooth if and only if $f \circ \varphi : \varphi(U \cap V) \to \mathbb{C}$ is smooth for any chart $\varphi : V \to \mathbb{C}$ with

$V \cap U \neq \varnothing$. Locally, the differential operators make also sense on the coordinate neighbourhoods of X, but of course ∂_x and ∂_y depend on the chart. However, the condition

$$\partial_x f(a) = \partial_y f(a) = 0 \tag{6}$$

is invariantly defined for a change of coordinates φ_{ij} is biholomorphic so that both $\partial_x \varphi_{ij}(a) = \partial_y \varphi_{ij}(a) \neq 0$. We let $\mathfrak{m}_a$ be the germ of smooth functions such that (6) holds. This is in fact a maximal ideal of $C^\infty_{X,a}$ the germs of smooth functions on X at a.

1. Definition (cotangent space). The quotient space

$$T_a^* X := \mathfrak{m}_a / \mathfrak{m}_a^2$$

is the **cotangent space of** X **at** a. It is a complex $\mathbb{C}$-vector space in a natural way. If U is an open neighbourhood of a and $f \in C^\infty_X(U)$, then the differential $d_a f \in T_a^* X$ is the element

$$d_a f := (f - f(a)) \bmod \mathfrak{m}_a^2$$

(note that $f - f(a) \in \mathfrak{m}_a$ for it obviously vanishes at a). In particular, $d_a c = 0$ for any constant function.

2. Proposition [Fo, 9.4]**.** *If $U \subset X$ is a chart with coordinate $z = x + iy$, then $(d_a x,\, da_y)$ as well as $(d_a z, d_a \bar{z})$ form a basis of T_a^X and for any smooth function defined near a, we have*

$$\begin{aligned} d_a f &= \partial_x f(a) d_a x + \partial_y f(a) d_a y \\ &= \partial_z f(a) d_a z + \partial_y f(a) d_a \bar{z}. \end{aligned}$$

Proof. We will carry out the proof for (x, y), the case $(z, \bar{z})$ working similarly.

Step 1. *$(d_a x, d_a y)$ is a basis of $T_a^* X$.* First of all, they generate $T_a^* X$ for if we expand $[f] \in T_a^* X$ for a smooth representative $f \in \mathfrak{m}_a$ into a Taylor series $f(x, y) = c_1(x - x(a)) + c_2(y - y(a)) + \tilde{f}$ for $c_{1,2} \in \mathbb{C}$, then $f(a) = \tilde{f} \in \mathfrak{m}_a^2$ so that taking the differential yieds $d_a f = c_1 d_a x + c_2 d_a y$. This is zero $\Leftrightarrow c_1(x - x(a)) + c_2(y - y(a)) = \sum g_i h_i$ with finitely many g_i, $h_i \in \mathfrak{m}_a^2$. Hence ∂_x and ∂_y evaluated at a implies $c_1 = 0$ and $c_2 = 0$.

Step 2. *Expresssion of $d_a f$ with respect to this basis.* If f is smooth near a, then its Taylor series gives

$$f - f(a) = \partial_x f(a)(x - a) + \partial_y f(a)(y - a) + \tilde{f}$$

with $\tilde{f} \in \mathfrak{m}_a^2$. Hence $d_a f = \partial_x f(a) d_a x + \partial_y f(a) d_a y$.

□

3. Cotangent vectors and their type. If (U, z) and $(U, \tilde{z})$ are two different coordinates around $a \in X$, then

$$c := \partial_z \tilde{z}(a) \in \mathbb{C}^* \quad \text{and} 0 = \partial_{\bar{z}} \tilde{z}(a) = 0.$$

Hence $d_a \tilde{z} = \partial_z \tilde{z}/z - a) + \text{terms in } \mathfrak{m}_a^2$ so that $d_a \tilde{z} = c d_a z$ and $d_a \bar{\tilde{z}} = \bar{c} d_a \bar{z}$. Hence both $d_a z$ and $d_a \tilde{z}$ as well as $d_a \bar{z}$ and $d_a \bar{\tilde{z}}$ span the same complex vector space

$$T_a^{1,0*} X := \mathbb{C} d_a z, \quad T_a^{0,1*} X := \mathbb{C} d_a \bar{z}.$$

In particular, we can decompose $d_a f$ into a $(1, 0)$- and $(0, 1)$-component denoted $\partial f(a)$ and $\bar{\partial} f(a)$ respectively, that is,

$$\partial f(a) = \partial_z f(a) d_a z \quad \text{and} \quad \bar{\partial} f(a) = \partial_{\bar{z}} f(a) d_a \bar{z}.$$

4. Definition (cotangent bundle and 1-forms). The **cotangent bundle** is the set $T^*X = \bigcup_{a\in X} T_a^*X$. We denote by $\pi : T^*X \to X$ the natural projection wich assigns to $\lambda \in T_a^*X$ its base point $\pi(\lambda) = a$. A 1-form over $U \subset X$ open is a section of T^*X, i.e. a map $\omega : U \to T^*X$ such that $\pi \circ \omega = \mathrm{Id}_U$. Similarly, we can define $T_a^{1,0*}X$ and $T_a^{0,1*}X$ with $(1,0)$ and $(0,1)$-forms as sections.

Note that any 1-form ω can be locally written as $\omega = fdx + gdy$ for functions $f,\ g : U \to \mathbb{C}$ defined on the coordinate neighbourhood U. Similarly, $(1,0)$- and $(0,1)$-forms can be written as fdz and $gd\bar{z}$.

5. Definition (smooth and holomorphic 1-forms). A 1-form ω is called **smooth** if locally $\omega = fdx + gdy$ for smooth functions f and g over U. A similar definition applies for smooth $(1,0)$ and $(0,1)$-forms. We denote by $\mathcal{A}_X^1(U)$, $\mathcal{A}^{1,0}(U)$ and $\mathcal{A}^{0,1}(X)$ the space of smooth 1-, $(1,0)$- and $(0,1)$-forms. Moreover, we call a 1-form **holomorphic** if locally $\omega = fdz$ for $f \in \mathcal{O}(U)$. We write $\Omega^1(U)$ for the holomorphic 1-forms over U. In particular, a holomorphic 1-form is a smooth $(1,0)$-form.

6. Examples. For every smooth function, $df(a) := d_af$, ∂f and $\bar{\partial} f$ are smooth 1-, $(1,0)$- and $(0,1)$-forms respectively. If $f \in \mathcal{O}(U)$, then $\partial f \in \Omega^1(U)$.

7. Remark. Note that we can multiply any smooth 1-form by a smooth function etc. so that $\mathcal{A}^1(U)$, $\mathcal{A}_X^{1,0}(U)$ and $\mathcal{A}_X^{0,1}(U)$ are C_X^∞-modules in a natural way. Similarly, $\Omega^1(U)$ is an $\mathcal{O}_X(U)$-module.

Recall from your linear algebra course the *exterior product* Λ^2V of a vector space which was generated by elements of the form $v_1 \wedge v_2, v_i \in V$, subject to the relations $(v_1 + v_2) \wedge v_3 = v_1 \wedge v_3 + v_2 \wedge v_3$, $(\lambda v_1) \wedge v_2 = \lambda(v_1 \wedge v_2)$ where λ is a scalar, and $v_1 \wedge v_2 = -v_2 \wedge v_1$. If $e_1, \ldots, e_n$ is a basis for V, then $e_i \wedge e_j$, $i < j$ is a basis for Λ^2V. Aapplying this pointwise we define the second exterior power of the cotangent bundle by

$$\Lambda^2T^*X := \bigcup_{a\in X} \Lambda^2T_a^*X.$$

A basis is given by $d_ax \wedge d_ay$ resp. $d_az \wedge d_a\bar{z}$. It follows that similar constructions such as $\Lambda^2T_X^{1,0*}$, $\Lambda^2T_X^{0,1*}$ must be trivial for $d_az \wedge d_az = 0$ etc. Now a (smooth) 2-form is a section of Λ^2T^*X locally of the form $\omega_a = f(a)d_ax \wedge d_ay = 2if(a)d_az \wedge d_a\bar{z}$ for f smooth. We denote the space of smooth 2-forms over U by $\mathcal{A}_X^2(U)$ or $\mathcal{A}_X^{1,1}(U)$. Note that $\Omega_X^2 = 0$. Note that we have a natural map

$$\mathcal{A}_X^1(U) \times \mathcal{A}_X^1(U) \to \mathcal{A}_X^2(U), \quad (\omega, \sigma) \mapsto \omega \wedge \sigma.$$

If locally, $\omega = fdx + gdy$ and $\sigma = \tilde{f}dx + \tilde{g}dy$, then$\omega \wedge \sigma = f\tilde{g} - g\tilde{f})dx \wedge dy$.

8. Remark. Since $\Lambda^0V = k$ the ground field of V we have $\Lambda^0T^*X = X \times \mathbb{C}$ so that $\mathcal{A}_X^0(U) \cong C_X^\infty(U)$, the smooth sections over U.

9. Exterior derivative of forms. We now extend the differential to map $d : \mathcal{A}_X^1(U) \to \mathcal{A}_X^2(U)$ for any open subset U of X, and similarly $\partial : \mathcal{A}_X^{0,1}(U) \to \mathcal{A}_X^{1,1}(U)$

and $\bar\partial : \mathcal{A}^{1,0}_X(U) \to \mathcal{A}^{1,1}_X(U)$. If locally $\omega = \sum_i f_i dg_i$ for functions f_i and differentials dg_i (we know such a representation to exist) we let

$$\begin{aligned} d\omega &= \sum df_i \wedge dg_i \\ \bar\partial\omega &= \sum \bar\partial f_i \wedge dg_i \\ \partial\omega &= \sum \partial f_i \wedge dg_i. \end{aligned}$$

For instance, if $\omega = f dz + g d\bar z$, then

$$\begin{aligned} d\omega &= (-\bar\partial f + \partial g) dz \wedge d\bar z \\ \bar\partial\omega &= -\bar\partial f dz \wedge d\bar z \\ \partial\omega &= \partial g dz \wedge d\bar z. \end{aligned}$$

A priori this depends on the coordinates $(z, \bar z)$ but it is straightforward to show that different coordinates give the same result (this follows essentially from the skew-symmetry of $\wedge$, see [Fo, 9.13]).

10. Elementary properties. *Let $f \in C^\infty_X(U)$ and $\omega \in \mathcal{A}^1_X(U)$. Then*

(i) *$d\omega = \partial\omega + \bar\partial\omega$. In particular, $\omega \in \Omega^1_X(U) \Leftrightarrow \bar\partial\omega = 0$.*
(ii) *$d \circ df = \partial \circ \partial f = \bar\partial \circ \bar\partial f = 0$. In particular, $\partial \circ \bar\partial f = -\bar\partial \circ \partial f$.*
(iii) *$d(f\omega) = df \wedge \omega + f d\omega$ etc.*

The proof is straightforward and a good exercise.

11. Definition (closed and exact differential forms). A differential form ω is **closed** if $d\omega = 0$, and **exact** if $\omega = d\sigma$.

12. Remark. In two dimensions it is elementary to see that a closed form is *locally* exact. For general smooth manifolds this is known as *Poincaré Lemma* (we are going to prove a $\bar\partial$-version of this in Theorem 2.41).

13. Proposition [Fo, 9.16]. *If $U \subset X$ open $\Rightarrow$*

(i) *every holomorphic 1-form is closed;*
(ii) *every closed 1-form in $\mathcal{A}^{1,0}_X(U)$ is holomorphic.*

Proof. If $\omega \in \Omega^1_X(U) \subset \mathcal{A}^{1,0}_X(U)$, then $\bar\partial\omega = 0$ for it is holomorphic, and $\partial\omega = 0$ for it is of type $(1,0)$. Therefore $d\omega = \bar\partial\omega + \partial\omega = 0$. Similarly, if $\omega \in \mathcal{A}^{1,0}_X(U)$, then $0 = d\omega = \bar\partial\omega$. ☐

14. Pull-back for differential forms. Finally, we discuss the pull-back of differential forms. Recall that a holomorphic map $F : X \to Y$ induced a map $F^*_V : \mathcal{O}_Y(V) \to \mathcal{O}_X(F^{-1}(V))$ for any $V \subset Y$ open by letting $F^*(f) = f \circ F$. Of course, requiring only smoothness of F gives merely a map $F^*_V : C^\infty_Y(V) \to C^\infty_X(F^{-1}(V))$. This can be generalised to a map

$$F^* \mathcal{A}^p_Y(U) \to \mathcal{A}^p_X(F^{-1}(V))$$

on differential forms as follows. Namely, if $f \in C^\infty_Y(V)$, then we define

$$F^* df := d(F^* f) = d(f \circ F)$$

and we extend F^* as an algebra morphism over all of $\mathcal{A}^*$, i.e.

$$F^*(\omega \wedge \tau) = F^*\omega \wedge F^*\tau.$$

It follows in particular that

$$F^* d\omega = d(F^*\omega).$$

With these rule we can compute the **pull-back** of $\omega \in \mathcal{A}^p_Y(V)$ for any ω.

15. Application: Integration of 2-forms. If $\omega \in \mathcal{A}^2_X(X)$ for a (compact Riemann) surface we want to make sense out of the expression $\int_X \omega$, the **integral** of ω over X. We first consider the case of a domain $X = U \subset \mathbb{C}$. We assume that

$$\operatorname{supp}\omega = \overline{\{z \in U \mid \omega_z \neq 0\}},$$

the so-called *support* of ω. If $\omega = f dx \wedge dy$, then $\operatorname{supp}\omega = \operatorname{supp} f$ where f is the usual suport of a function. Since the differential of a biholomorphic map never vanishes, this is indeed independent of the concrete coordinate system we use to express ω. We now define

$$\int_U \omega := \int_U f(x,y) dxdy,$$

where $dxdy$ denotes the Lebesgue measure of U under the identification with an open subset of $\mathbb{R}^2$ using the coordinates x and y. Now if $\varphi(x,y) = u(x,y) + iv(x,y) : V \to U$ is a biholomorphic map, then its functional determinant is $\det \mathrm{Jac}\varphi = \partial_x u \partial_y v - \partial_y u \partial_x v = |\varphi'|^2$, the latter in view of the Cauchy-Riemann equations. By the usual transformation formula for integrals, we have

$$\int_U f dxdy = \int_V (f \circ \varphi)|\varphi'|^2 dxdy.$$

On the other hand, writing $\omega = if/2dz \wedge d\bar{z}$, we have $\varphi^*\omega = i(f\circ\varphi)/2d\varphi \wedge d\bar{\varphi} = |\varphi'|^2\omega$ we obtain

$$\int_U \omega = \int_{\varphi(V)} \omega = \int_V \varphi^*\omega, \tag{7}$$

that is, the number $\int_U \omega$ is invariant under change of holomorphic variables (more generally, one can show that it is invariant under *oriented* diffeomorphisms so integration is really a feature of the underlying differentiable surface).

Next consider a chart $\psi : U \to V$ of the Riemann surface X. If $\omega \in \mathcal{A}^2_X(X)$ is a 2-form with support in U we define

$$\int_X \omega := \int_U \omega := \int_V \psi^{-1*}\omega.$$

It follows from (7) that this is indeed independent of the given chart, since any other chart gives rise to a biholomorphic transition function φ which does not change the value of $\int_U \omega$.

Finally, let $\omega \in \mathcal{A}^2_X(X)$ be a general 2-form with copmact support (this is automatic if X is compact and ensures that the integral is finite). Cover X by coordinate charts U_k and take a partition of unity $\{f_k\}$ subordinate to $\{U_k\}$ (see the introduction of Section 1 for the notion of a partition of unity). Then $\omega = \sum(f_k\omega)$ is the locally finite, hence well-defined sum of differential forms $\omega_k = f_k\omega$ with compact support in the chart domain U_k. We then define

$$\int_X \omega := \sum_k \int_{U_k} (f_k\omega).$$

A tedious, but straight forward computation shows that this is indeed well defind, that is, independent of the partition of unity $\{(U_k, f_k)\}$. This defines the integral of ω which has the following properties:

(i) *Linearity.* If $\lambda \in \mathbb{C}$ and ω, $\tau \in \mathcal{A}^2(X)$ are 2-forms of compact support, then

$$\int_X (\lambda\omega + \tau) = \lambda \int_X \omega + \int_X \tau.$$

Again, this is a direct verification.

(ii) *Stokes' Theorem.* Assume that X is compact nad $\omega \in \mathcal{A}^1(X)$. Then

$$\int_X d\omega = 0.$$

This can be reduced to the case $\int_U d\omega$, where $U \subset \mathbb{C}$ is an open set with compact closure and smooth boundary ∂U. For instance, if $U = \overline{D_R \backslash D_r}$ for two discs with $R > r$, then

$$\int_U d\omega = \int_{D_R} \omega|_{D_R} - \int D_r \omega|_{D_r} \tag{8}$$

with the obvious restriction of 1-forms. By passing to radial coordinates one can directly compute the integral, see [Fo, 10.19-20].

(iii) *Poincaré-Pairing.* Let X be compact. The bilinear map

$$\mathcal{A}^1(X) \to \mathcal{A}^1(X) \to \mathbb{C}, \quad (\omega, \tau) \mapsto \int_X \omega \wedge \tau$$

is non-degenerate, that is, if $\int_X \omega \wedge \tau = 0$ for all $\tau \in \mathcal{A}^1(X) \Rightarrow \omega = 0$ (Exercise).

Sheaves. The objects $\mathcal{A}0$, Ω^p etc wich are associated with a Riemann surfaces are examples of *sheaves* a concept we investigate next. Sheave and their cohomology will be a powerful tool in the investigation of Riemann surfaces, but they are actually important in many areas of geometry. The following definition is completely general and valid for any topological space X. Soon, however, we are going to specialise to the case of a Riemann surface.

16. Definition (sheaf). Let X be a topological space. A **sheaf** (of abelian groups) $\mathcal{F}$ over X is a topological space together with a continuous map $\pi : \mathcal{F} \to X$, the **projection** such that

(i) π is a local homeomorphism;
(ii) for each point $p \in X$, the **stalk** $\mathcal{F}_p := \pi^{-1}(p)$ is an abelian group;
(iii) the group operations are continuous. Concretely, consider $\pi^*\mathcal{F} := \{(s,t) \mid \pi(s) = \pi(t)\} \subset \mathcal{F} \times \mathcal{F}$ together with the induced topology. Then the assignement $(s,t) \in \pi^*\mathcal{F} \mapsto s - t \in \mathcal{F}$ is continuous.

A **sheaf map** $F : \mathcal{F} \to \mathcal{G}$ between sheaves $\pi : \mathcal{F} \to X$ and $\tilde{\pi} : \mathcal{G} \to X$ is a continuous map such that $\tilde{\pi} \circ F = \pi$. In particular, any sheaf map is stalk preserving, i.e. $F(\mathcal{F}_p) \subset \mathcal{G}_p$. A **sheaf morphism** is a sheaf map which is also a group morphism on any stalk.

17. Remark. In the same way, we can consider sheaves of rings, fields, vector spaces etc.

18. Examples.

(i) Let G be an abelian group which we equip with the discrete topology – every set is open. Then $\mathcal{G} = X \times G$ together with the product topology and $\pi =$ projection on the first factor, is a sheaf: $U_g := U \times \{g\}$, $U \subset X$ open and $g \in G$ is a basis for the topology, and π restricted to U_g is obviously a homeomorphism. Note that this might not be the "natural" topology for groups such as $\mathbb{C}$ (In case of a Riemann surface, $E = X \times \mathbb{C}$ for X a Riemann surface and with the product topology with respect to the standard Euclidean topology on $\mathbb{C}$, E would be the trivial vector bundle, and not a constant sheaf.)
(ii) Let X be a Riemann surface. Then $\pi : |\mathcal{O}| \to X$ is a sheaf, cf. Section 1.1.3.

In view to understand the last example better it is instructive to consider the topology of a sheaf $\mathcal{F} \to X$ from a different perspective. A **section** of $\mathcal{F}$ over $u \subset X$ open is a continuous map $\sigma : U \to \mathcal{F}$ such that $\pi \circ \sigma = \mathrm{Id}_U$. In particular, $\sigma(p) \in \mathcal{F}_p$. We denote the set of sections over U by $\Gamma(U, \mathcal{F})$. Now pick any $s \in \mathcal{F}$ and an open neighbourhood V of s in $\mathcal{F}$ such that $\pi|_V$ is a homeomorphism onto its image. It follows that $\sigma := (\pi|_V)^{-1}$ is a section whose image is an open neighbourhood of s. In fact, if τ is any other section with $\tau(p) = s$ taking a suitably small neighourhood V of s shows that $\tau_0 = \tau|_{\tau^{-1}(V)}$ takes values in V so that $\tau_0 = (\pi|_V)^{-1}$, that is, any section is locally of this form. In particular, any section σ is an *open* map, and the images of sections form a basis of the topology. Furthermore, if σ, $\tau \in \Gamma(U, \mathcal{F})$, then compounding with the continuous group operations gives a continuous section $f - g \in \Gamma(U, \mathcal{F})$, that is, $\Gamma(U, \mathcal{F})$ defines a group in a natural way. In particular, for any U there exists $\sigma_0 \in \gamma(U, \mathcal{F})$ defined by $\sigma_0(p) = 0_p$ = the zero element in $\mathcal{F}_p$, the so-called **zero section**. Note that in passing, the fact that π is a local homeomorphism garantuees local existence of non-trivial sections. In general, however, $\Gamma(X, \mathcal{F})$, the group of global sections, might consist of the zero section only.

If $F : \mathcal{F} \to \mathcal{G}$ is a sheaf map, then we have an induced map $F_* : \Gamma(U, \mathcal{F}) \to \Gamma(U, \mathcal{G})$ defined by $F_*\sigma = F \circ \sigma$ for F is continuous. In particular, a sheaf map is open and a local homeomorphism for the sections generate the topology. If, furthermore, F is a sheaf morphism, then F_* is also a group morphism.

Now with a sheaf we have associated the family of groups $\Gamma(U, \mathcal{F})$. To what extent does this determine the sheaf $\mathcal{F}$? We first axiomatise the properties of $\Gamma(U, \mathcal{F})$.

19. Definition (presheaf). A **presheaf** (of abelian groups) over X is an assignment $U \mapsto \mathcal{F}_U$ of abelian groups for any open set U of X, together with morphisms $\rho_{UV} : \mathcal{F}_U \to \mathcal{F}_V$, the so-called **restriction maps**, for any pair $V \subset U$ of open subsets of X, which satisfy

- $F_\varnothing = 0$;
- $\rho_{UU} = \mathrm{Id}_{\mathcal{F}_U}$;
- $\rho_{VW} \circ \rho_{UV} = \rho_{UW}$ for any triple $W \subset V \subset U$ of open subsets of X.

We sometimes also write $g|_V$ for $\rho_{UV}(g)$ if no confusion arises. A **presheaf map** is a family of maps $F_U : \mathcal{F}_U \to \mathcal{G}_U$ which commutes with restrictions, i.e. $\rho^{\mathcal{G}}_{UV} \circ F_U = F_V \circ \rho^{\mathcal{F}}_{UV}$. It is a **presheaf morphism** if it is group morphism for any open subset U of X.

20. Examples.

(i) Let G be an abelian group. Then $\mathcal{F}_U = G$ for $U \neq \varnothing$ and $\rho_{UV} = \mathrm{Id}_G$ is a presheaf, the so-called **constant presheaf** associated with G which we denote by $\mathbf{G}$.

(ii) Let X be a Riemann surface. Then $\mathcal{O}_X(U)$ = holomorphic functions $U \to \mathbb{C}$ defines a presheaf together with the usual restriction maps.

(iii) For any sheaf $\mathcal{F}$, $\mathcal{F}_U := \Gamma(U, \mathfrak{F})$ defines a presheaf, the **presheaf of sections** of $\mathcal{F}$.

21. Sheafification. Conversely, consider a presheaf $\mathcal{F}_U$. Then there is a naturally associated sheaf $\hat{\mathfrak{F}}$. We construct $\hat{\mathcal{F}}$ as the union of "stalks" $\mathcal{F}_p$ for $p \in X$. Towards that end, let $\mathcal{U}(p) = \{U \subset X \text{ open} \mid p \in U\}$. Then

$$\mathcal{F}_p := \varinjlim_{U \in \mathcal{U}(p)} \mathcal{F}_U,$$

that is, the stalk is the direct limit of $\mathcal{F}_U$ which inherits a natural structure as an abelian group. By definition, an element $s \in \mathcal{F}_p$ is an equivalence class represented by an element $g \in \mathcal{F}_U$ for some $U \in \mathcal{U}(p)$, with elements $g \in \mathcal{F}_U$ and $h \in \mathcal{F}_V$ for U, $V \in \mathcal{U}(p)$ identified if there exists an open subset $W \subset U \cap V$ in $\mathcal{U}(p)$ with $\rho_{UW}(g) = \rho_{VW}(h)$ (the "germ" g_p of g). This defines $\hat{\mathcal{F}} = \bigcup_{p \in X} \mathcal{F}_p$ as a set, together with π = projection on the base point. Note that we get maps $\rho_{Up} : \mathcal{F}_U \to \mathcal{F}_p$, $\rho_{Up}(g) = g_p$ the germ of g at $p \in U$. As a base of topology we take the images of *locally constant sections* $\sigma : U \to \hat{\mathcal{F}}$, that is, locally $\sigma(p) = g_p$ for some $g \in \mathcal{F}_V$, $V \subset U$. It is easy to check that this defines indeed a base of topology [Gu, Section 2.b] for which π is clearly a local homeomorphism with stalks $\pi^{-1}(p) = \mathcal{F}_p$ and local sections $\Gamma(U, \hat{\mathcal{F}}) = \{\sigma : U \to \mathcal{F} \mid \sigma \text{ locally constant}\}$. Further, it is not difficult to see that the group operation is continuous, cf. again [Gu, Section 2.b].

22. Examples.

(i) Let X be a Riemann surface and consider the presheaf $\mathcal{O}_X(U)$. The associated stalks $\mathcal{O}_{X,p}$, $p \in X$, are just germs of holomorphic functions (fixing a local uniformising coordinate z these can be identified with $\mathbb{C}\{z\}$, the ring of convergent power series in z). Conversely, the presheaf of sections of $|\mathcal{O}_X|$ is just $\mathcal{O}_X(U)$.

(ii) Let $\mathbf{G}$ be the constant presheaf associated with the abelian group G. It follows that the local sections $\sigma \in \Gamma(U, \hat{\mathbf{G}})$ which are locally constant are just the continuous maps $\sigma : U \to G$, where G is endowed with the discrete topology. It follows that if U is an open set with n connected components, then $\Gamma(U, \hat{\mathbf{G}}) \cong G^n$. In particular, the sheaf of sections of $\hat{\mathbf{G}}$ is not isomorphic to $\mathbf{G}$.

It is clearly in order to characterise those presheaves which arise as presheaves of sections.

23. Definition (complete presheaf). A presheaf $\mathcal{F}$ on X is called **complete** if for any open covering $\{V_i\}$ of an open subset U of X, the following conditions hold:

(i) If $s \in \mathcal{F}(U)$ is such that $s|_{V_i} = 0 \in \mathcal{F}(V_i)$ for all i, then $s = 0$ in $\mathcal{F}(U)$ ("s is determined by restriction to open subsets", "local injectivity").

(ii) If there exists $s_i \in \mathcal{F}(V_i)$ for each i such that $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$, then there exists $s \in \mathcal{F}(U)$ such that $s|_{V_i} = s_i$ ("local compatible sections can be glued together", "local surjectivity").

24. Examples.

(i) If X is a Riemann surface, then the presheaf of holomorphic functions $\mathcal{O}_X(U)$ is a complete presheaf. Similarly, we can consider the complete presheaves
- $\mathcal{O}^*_X(U)$ = nowhere vanishing holomorphic functions $U \to \mathbb{C}$ (modelled on the multiplicative group $\mathcal{O}_X(U)^*$)
- $\mathcal{M}_X$ = meromorphic functions, $\mathcal{M}^*_X$ = meromorphic functions not identically zero
- Ω^p_X = sheaf of holomorphic p-forms

- the "smooth sheaves" C^∞ of smooth functions or $\mathcal{A}^p$ of smooth p-forms etc.

(ii) $\mathbf{G}$ is not complete for if $U = U_0 \cup U_1$ has two (disjoint) connected components, the local sections $g_i \in \mathbf{G}(U_i)$ do not glue to a section in $\mathbf{G}(U)$ unless $g_0 = g_1$.

25. Proposition [Gu, Lemma 3]**.** *A presheaf $\mathcal{F}_U$ arises as a sheaf of sections $\Leftrightarrow$ $\mathcal{F}_U$ is complete. In fact, $\mathcal{F}_U$ is complete $\Leftrightarrow$ $\mathcal{F}_U \cong \Gamma(U, \hat{\mathcal{F}})$, where $\hat{\mathcal{F}}$ is the associated sheaf. In particular, we can identify sheaves and complete presheaves in a natural way, and we will therefore refere to a complete presheaf simply as a sheaf.*

Proof. Only the converse requires proof. So assume that $\mathcal{F}_U$ is a complete presheaf, and consider the sheaf $\hat{\mathcal{F}}$ associated with $\mathcal{F}_U$. We show that the natural group morphism $\gamma_U : \mathcal{F}_U \to \Gamma(U, \hat{\mathcal{F}})$ which associates with $g \in \mathcal{F}_U$ the section $\gamma_U(g)(p) = g_p$, is a bijection.

γ is injective. Assume that $g_p = 0_p$ for all $p \in U$. This means that locally, the section $\gamma_U(g)$ restricts to the zero section. By the local injectivity property we conclude $\gamma(g) \equiv 0$.

γ is surjective. Let $\sigma \in \Gamma(U, \hat{\mathcal{F}})$. Then locally, $\sigma|_{U_\alpha} = \gamma_{U_\alpha}(g_\alpha)$ for a suitable cover $U = \bigcup U_\alpha$. Since $g_\alpha|_{U_\alpha \cap U_\beta}$ and $g_\beta|_{U_\alpha \cap U_\beta}$ map to $\sigma|_{U_\alpha \cap U_\beta}$ under $\gamma_{U_\alpha \cap U_\beta}$ we have $g_\alpha|_{U_\alpha \cap U_\beta} = g_\beta|_{U_\alpha \cap U_\beta}$ by the injectivity just established. Hence there exists $g \in \mathcal{F}_U$ which restricts to g_α over U_α by the local surjectivity property. □

26. Remark. In view of the local nature of a complete presheaf $\mathcal{F}_U$, it is already determined by $\mathcal{F}_{U_\alpha}$ for a base $\{U_\alpha\}$ of the topology of X. Indeed, we have

$$\mathcal{F}_U = \varprojlim_{U_\alpha \subset U} F_{U_\alpha},$$

where $\varprojlim$ denotes the *projective limit* of the partially ordered set $\{U_\alpha \mid U_\alpha \subset U\}$. This is the subset $\{(g_\alpha)_\alpha \in \prod_\alpha U_\alpha \mid g_\alpha = \rho_{U_\beta U_\alpha} g_\beta \text{ if } U_\alpha \subset U_\beta\}$. Therefore we will often only specify complete presheaves for a base of topology. Similarly, it is enough to define presheaf morphism for $\mathcal{F}(U_\alpha) \to \mathcal{G}(U_\alpha)$. We leave the details as an Exercise.

27. Further types of sheaves. Let $\pi : \mathcal{F} \to X$ be a sheaf.

(i) *Restriction of a sheaf.* If $E \subset X$ is any subset we call $\mathcal{F}|_E := \pi^{-1}(E)$ the **restriction** of $\mathcal{F}$. For $p \in E$ we have $(\mathcal{F}|_E)_p = \mathcal{F}_p$. For instance, if $U \subset X$ is a connected *open* subset of a Riemann surface X we can consider the restriction $\mathcal{O}_X|_U$ which is just the sheaf of holomorphic germs of the Riemann surface U. (If U is not open, for instance, $U = \{p\}$ is a point, then $\mathcal{O}_X|_E$ is in general not even a subsheaf of continuous functions $C^0_E = \bigcup_{p \in X} C^0_{E,p}$ over E, for instance $\mathcal{O}_X|_{\{p\}} = \mathcal{O}_{X,p} \cong \mathbb{C}\{z\}$, but $C^0_{\{p\}} = \mathbb{C}$.)

(ii) *Subsheaves.* Let $\mathcal{G} \subset \mathcal{F}$ be an open subset. Then $\mathcal{G}$ is a subsheaf of $\mathcal{F}$ if for all $p \in X$, $(\pi|_{\mathcal{G}})^{-1}(p) = \mathcal{G}_p = \mathcal{G} \cap \mathcal{F}_p$ is a subgroup of $\mathcal{F}_p$. Projection is the restriction of π to $\mathcal{G}$.

(iii) *Quotient sheaves.* If $\mathcal{G} \to X$ is a subsheaf of $\mathcal{F}$ we can define the **quotient sheaf** $\mathcal{Q} = \mathcal{F}/\mathcal{G}$ as follows. The stalks $\mathcal{Q}_p$ are just the quotient groups $\mathcal{F}_p/\mathcal{G}_p$. We let $Q = \bigcup_p \mathcal{Q}_p$ and take the natural projection to X. We have a natural quotient map $q : \mathcal{F} \to \mathcal{Q}$ by projecting any stalk to the quotient $\mathcal{F}_p/\mathcal{G}_p$. A set $U \subset \mathcal{Q}$ is by definition open if $q^{-1}(U) \subset \mathcal{F}$ is open. It is easy to see that this gives $\mathcal{Q}$ the structure of a sheaf. It follows that a section $\sigma : U \to \mathcal{Q}$ can be locally lifted to a section $\tilde{\sigma} : U \to \mathcal{F}$ with $q \circ \tilde{\sigma} = \sigma$.

28. Example (skyscraper sheaf). Let X be a Riemann surface with $\mathcal{O}_X$ the sheaf of holomorphic functions. Choose a point $p \in X$ and consider the subsheaf

$$\mathcal{S}(U) = \{f \in \mathcal{O}_X(U) \mid f(p) = 0 \text{ whenever } x \in U\}.$$

This is indeed a subsheaf for $\mathcal{S}$ is open in $\mathcal{O}_X$ (the sections of $\mathcal{S}$ which form a basis for the topology of $\mathcal{S}$ are also open in $\mathcal{O}_X$) and $\mathcal{S}_p = \mathcal{O}_{X,p} \cap \mathcal{S}$ is clearly a subgroup (actually the maximal ideal) of $\mathcal{O}_{X,p}$. The quotient sheaf $\mathcal{Q} = \mathcal{O}_X/\mathcal{S}$ is then the union of stalks $\mathcal{Q}_a = \{0\}$ if $a \neq p$ and $\mathcal{Q}_p = \mathcal{O}_{X,p}/\mathcal{S}_p = \mathcal{O}_{X,p}/\mathfrak{m} = \mathbb{C}$. We can extende this construction to any discrete set of points. The resulting sheaf is called a skyscraper sheaf for obvious reasons.

29. Exact sequences. Let $F : \mathcal{F} \to \mathcal{G}$ be a sheaf homomorphism. Then we let

$$\ker F = \bigcup_{a \in X} \ker F_a = F^{-1}(0)$$

and

$$\operatorname{im} F = \bigcup_{a \in X} \operatorname{im} F_a$$

be the **kernel** and the **image sheaf** respectively. In terms of presheaves, these are the sheaves associated with the presheaves $\mathcal{F}_U = \ker F_U$ and $\mathcal{G}_U = \operatorname{im} F_U$. It follows that $\operatorname{im} F \cong \mathcal{F}/\ker F$. Note in passing that while the kernel presheaf is already complete, the image sheaf is not. This enables us to define an **exact sequence of sheaves**

$$\mathcal{F} \xrightarrow{F} \mathcal{G} \xrightarrow{G} Q$$

which by definition means that $\operatorname{im} F \cong \ker G$ as sheaves. In particular, if we let $\mathbf{0}$ denote the trivial sheaf with stalks the zero group, a sequence

$$\mathbf{0} \longrightarrow \mathcal{F} \xrightarrow{F} \mathcal{G} \xrightarrow{G} Q \longrightarrow \mathbf{0}$$

is exact $\Leftrightarrow$ F is an **injection**, that is, F is an isomorphism onto a subsheaf of $\mathcal{G}$, and G is a **projection**, that is a sheaf morphism whose image is all of Q. Hence such a so-called *short exact sequence* is equivalent to

$$\mathbf{0} \longrightarrow \mathcal{S} \xrightarrow{\iota} \mathcal{F} \xrightarrow{\pi} F/\mathcal{S} \longrightarrow \mathbf{0},$$

where $\mathcal{S}$ is a subsheaf of $\mathcal{F}$, and ι and π are the natural injection and projection maps.

A prime example of such a short exact sequence is the **exponential sequence** of a Riemann surface given by

$$\mathbf{0} \longrightarrow \mathbb{Z} \xrightarrow{\iota} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \longrightarrow \mathbf{0},$$

where $\mathbb{Z}$ denotes the constant sheaf associated with $\mathbb{Z}$, and where exp is the holomorphic exponential map which maps a germ $[U, f]$ to $[U, \exp f]$ (thinking of U as a domain in $\mathbb{C}$ via a chart). Since we can always choose U to be simply-connected, we can take logarithms of nowhere vanishing holomorphic functions over U so that exp is indeed surjective. On the other hand, the sequence on presheaf level

$$\mathbf{0} \longrightarrow \mathbb{Z}_U \xrightarrow{\iota} \mathcal{O}_{X,U} \xrightarrow{\exp} \mathcal{O}_{X,U}^* \longrightarrow \mathbf{0},$$

is not exact for general U for it fails to be surjective (ultimately, this reflects the fact that the presheaf $\operatorname{im} F_U$ is not complete).

Cohomology. One of the principal uses of sheaves is to consider their associated *cohomology theory*. This gives rise to natural invariants of the underlying space. Again, the formalism applies to any topological space X, but of course, we are mainly interested in the case of a Riemann surface.

We start with some definitions for a general topological space X. Let $\mathcal{U} = \{U_\alpha\}$ an open covering of X. Further, let $N(\mathcal{U})$ be the **nerve of** $\mathcal{U}$, which we define as follows. The elements U_α of $\mathcal{U}$ are called the **vertices**. Any choice of $q+1$ subsets $U_0, \ldots U_q$ span a q**-simplex** $\sigma = (U_0, \alpha, U_q)$. The *open* set $U_0 \cap \ldots \cap U_q = |\sigma|$ is called the **support** of the simplex σ. Then the nerve $N(\mathcal{U})$ is the set of all q-simpleces, $q \geqslant 0$.

Next let $\mathcal{F} \to X$ be a sheaf. A q**-cochain of** $\mathcal{U}$ **with coefficients in the sheaf** $\mathcal{F}$ is a function f which assigns to every q-simplex in $N(\mathcal{U})$ a section $f(\sigma) \in \Gamma(|\sigma|, \mathcal{F})$. We denote the set of q-cochains by $C^q(\mathcal{U}, \mathcal{F})$, so

$$C^0(\mathcal{U}, \mathcal{F}) = \prod_\alpha \mathcal{F}(U_\alpha)$$
$$C^1(\mathcal{U}, \mathcal{F}) = \prod_{\alpha \neq \beta} \mathcal{F}(U_\alpha \cap U_\beta)$$
$$\ldots$$

This set inherits the natural algebraic structure of $\mathcal{F}$, so if $\mathcal{F}$ is a sheaf of abelian groups, $(f+g)(\sigma) = f(\sigma) + g(\sigma) \in \Gamma(|\sigma|, \mathcal{F})$. We define a group morphism

$$\delta^q : C^q(\mathcal{U}, \mathcal{F}) \to C^{q+1}(\mathcal{U}, \mathcal{F}),$$

the so-called **coboundary operator**, for $f \in C^q(\mathcal{U}, \mathcal{F})$ and $\sigma = (U_0, \ldots, U_{q+1}) \in N(\mathcal{U})$ by

$$\delta^q(f)(\sigma) = \sum_{i=0}^{q+1} (-1)^i \rho_{i|\sigma|}\big(f(U_0, U_{i-1}, U_{i+1}, \ldots, U_{q+1})\big) \in \Gamma(U_0 \cap \ldots \cap U_{q+1}, \mathcal{F}),$$

where $\rho_{i|\sigma|}$ denotes the restriction map from $\Gamma(U_0 \cap U_{i-1} \cap U_{i+1} \ldots \cap U_{q+1}, \mathfrak{C})$ to $\Gamma(|\sigma|, \mathcal{F})$. Then $C^q(\mathcal{U}; \mathcal{F})$ becomes a **(differential) complex**, i.e.

$$\delta^{q+1} \circ \delta^q = 0,$$

which is a straightforward, if tedious, computation. For sake of simplicity we often write δ instead of δ^q. Next we consider the subgroups

$$Z^q(\mathcal{U}, \mathcal{F}) = \{f \in C^q(\mathcal{U}, \mathcal{F}) \mid \delta f = 0\} = \ker \delta^q,$$

the q**-cocycles**, and

$$B^q(\mathcal{U}, \mathcal{F}) = \delta^{q-1} C^{q-1}(\mathcal{U},) = \operatorname{im} \delta^{q-1},$$

the so-called q**-coboundaries**. Since $\delta^2 = 0$, $B^q \subset Z^q$, and the quotient group

$$H^q(\mathcal{U}, \mathcal{F}) = \begin{cases} Z^q(\mathcal{U}, \mathcal{F})/B^q(\mathcal{U}, \mathcal{F}), & q > 0 \\ Z^0(\mathcal{U}, \mathfrak{F}), & q = 0 \end{cases}$$

is the q**-th cohomology group of** $\mathcal{U}$ **with coefficients in the sheaf** $\mathcal{F}$. These cohomology groups obviously depend on the covering $\mathcal{U}$, so we still need to work in order to turn this into an invariant of the underlying topological space. For H^0 ths is easy.

30. Lemma (0-th cohomology and global sections) [Gu, Lemma 4]**.** *For any covering* $\mathcal{U}$ *of* X *we have*

$$H^0(\mathcal{U}, \mathcal{F}) = \Gamma(X, \mathcal{F}).$$

Proof. A zero-cochain $f \in C^0(\mathcal{U}, \mathcal{F})$ assigns to each $U \in \mathcal{U}$ a section $f(U) \in \Gamma(U, \mathcal{F})$. By definition, $f \in H^0(X, \mathcal{F}) \Leftrightarrow \delta f = 0$. If we let $U_{\alpha\beta} := U_\alpha \cap U_\beta$ denote pairwise intersections for U_α and U_β in $\mathcal{U}$, the latter condition means that

$$\delta f(U_{\alpha\beta}) = f(U_\alpha)|_{U_{\alpha\beta}} - f(U_\beta)|_{U_{\alpha\beta}} = 0,$$

that is, if $U_{\alpha\beta} \neq \varnothing$ then the local sections $f(U_{\alpha\beta}) \in \Gamma(U_{\alpha\beta})$ agree on intersections and there exists a global section $\hat{f} \in \Gamma(X, \mathcal{F})$ which restricts to $f(U_\alpha)$. Conversely, a global section $\hat{f} \in \Gamma(X, \mathcal{F})$ obviously produces local sections $f(U_\alpha) = \hat{f}|_{U_\alpha}$ in $\Gamma(U_\alpha)$ which agree on the overlaps. □

Next we investigate H^q for $q > 0$ and various coverings. We call a covering $\mathcal{V} = \{V_a\}$ a **refinement** of $\mathcal{U} = \{U_\alpha\}$ if there exists a mapping $\mu : \mathcal{V} \to \mathcal{U}$ such that $V_a \subset \mu(V_a)$ for all $V_a \in \mathcal{V}$. Put differently, any vertex of $\mathcal{V}$ must sit inside some vertex of $\mathcal{U}$. The map μ is called the **refining map**. It induces a map

$$\mu : C^q(\mathcal{U}, \mathcal{F}) \to C^q(\mathcal{V}, \mathcal{F})$$

as follows. If $f \in C^q(\mathcal{U}, \mathcal{F})$ and $\tau = (V_0, \ldots, V_q)$ is a q-simplex in $N(\mathcal{V})$, then $\mu(f)(V_0, \ldots, V_q) = f(\mu(V_0), \ldots, \mu(V_q))|_{|\tau|}$. Note that $\varnothing \neq V_0 \cap \ldots \cap V_q \subset \mu(V_0) \cap \ldots \cap \mu(V_q)$ so that $(\mu(V_0), \ldots, \mu(V_q)$ is a q-simplex of $N(\mathcal{U})$. Clearly, μ is a group morphism and commutes with δ, i.e. $\mu \circ \delta = \delta \circ \mu$. It therefore descends to a group morphism

$$\mu^* : H^q(\mathcal{U}, \mathcal{F}) \to H^q(\mathcal{V}, \mathcal{F}).$$

Although a refinement map is not uniquely determined, its induced map at cohomology level is:

31. Lemma [Gu, Lemma 5]**.** *If $\mathcal{V}$ is a refinement of $\mathcal{U}$, and if $\mu : \mathcal{V} \to \mathcal{U}$ and $\nu : \mathcal{V} \to \mathcal{U}$ are two refining maps $\Rightarrow \mu^* = \nu^*$.*

Proof. Let $q = 0$. An element $f \in H^0(\mathcal{U}, \mathcal{F})$ is a collection $\{f(U_\alpha)\}$ such that $f(U_\alpha)|_{U_{\alpha\beta}} = f(U_\beta)|_{U_{\alpha\beta}}$. Hence $\mu(f)$ is the collection $\{f(\mu(V_a))\}$. Under the identification with global sections, both $\{f(U_\alpha)\}$ and $\{f(\mu(V_a))\}$ glue to the same global section, and similarly for ν. Hence $\mu^* = \nu^* = \mathrm{Id}$.

Let $q > 0$. We need to show that if $f \in Z^q(\mathcal{U}, \mathcal{F})$, then $\nu(f) - \mu(f) = \delta\theta(f)$ for some $\theta(f) \in C^{q-1}(\mathcal{V}, \mathcal{F})$. Modulo coboundaries, this means that $\nu = \mu$, i.e. $\nu^* = \mu^*$. We define $\theta : C^q(\mathcal{U}, \mathcal{F}) \to C^{q+1}(\mathcal{V}, \mathcal{F})$ as follows. If $f \in C^q(\mathcal{U}, \mathcal{F})$ and $\tau = (V_0, \ldots, V_{q-1}) \in N(\mathcal{V})$, then

$$\theta(f)(V_0, \ldots, V_{q-1}) = \sum_{j=0}^{q-1} (-1)^j f(\mu(V_0), \ldots, \mu(V_j), \nu(V_j), \ldots, \nu(V_{q-1}))|_{|\tau|}.$$

Now this has at least on μ- and one ν-entry in every summand. Taking the differential, a short computation on $\tau = (V_0, \ldots, V_q)$ shows that

$$\begin{aligned}\delta\theta(f)(V_0, \ldots, V_q) = &\sum_{j=0}^{q} (-1)^{j+1} \delta f(\mu(V_0), \ldots, \mu(V_j), \nu(V_j), \ldots, \nu(V_q))|_{|\tau|} \\ &+ \nu^*(f)(\tau) - \mu^*(f)(\tau),\end{aligned}$$

whence the assertion if $\delta f = 0$. □

Now we can define a partial ordering on the set of coverings as follows. We write $\mathcal{V} \leqslant \mathcal{U}$ if $\mathcal{V}$ is a refinement of $\mathcal{U}$. By the previous lemma there is a well-defined map $\rho_{\mathcal{U}\mathcal{V}} : H^q(\mathcal{U}, \mathcal{F}) \to H^q((\mathcal{V}, \mathcal{F})$ which is transitive, i.e. $\rho_{\mathcal{V}\mathcal{W}} \circ \rho_{\mathcal{U}\mathcal{V}} = \rho_{\mathcal{U}\mathcal{W}}$, and such that $\rho_{\mathcal{U}\mathcal{U}} = 0$. Note that the set of coverings is *directed*, that is, for any two coverings $\mathcal{U}$ and $\mathcal{V}$ one can find a covering $\mathcal{W}$ such that $\mathcal{W} \leqslant \mathcal{U}$ and $\mathcal{W} \leqslant \mathcal{V}$ (take for instance as vertices in $\mathcal{W}$ the intersections of the vertices in $\mathcal{U}$ and $\mathcal{V}$). We can therefore define

$$H^q(X, \mathcal{F}) = \varinjlim_{\mathcal{U}} H^q(\mathcal{U}, \mathcal{F})$$

which by definition is the group obtained by taking the product $\bigoplus_{\mathcal{U}} H^q(\mathcal{U}, \mathcal{F})$ and by identifying two elements $f \in H^q(\mathcal{U}, \mathcal{F})$ and $g \in H^q(\mathcal{V}, \mathcal{F})$ if there exists a common refinement $\mathcal{W}$ of $\mathcal{U}$ and $\mathcal{V}$ such that the images of f and g in $H^q(\mathcal{W}, \mathcal{F})$ agree. In particular, for each covering $\mathcal{U}$ there is a natural map $H^q(\mathcal{U}, \mathcal{F}) \to H^q(X, \mathcal{F})$. The cohomology thus obtained is usually referred to as **Čhech cohomology**. If we wish to distinguish it from other cohomology theories we sometimes write $\check{H}^q$ instead of H for emphasis.

32. General definition of direct limits. More generally, we can define the direct limit of groups as follows. Let $\{G_i\}_{i\in I}$ be a family of groups indexed by a directed set I, i.e. we have a partial ordering $\leqslant$ and for any two elements i and j in I there exists $k \in I$ such that $k \leqslant i, j$. Furthermore, we assume that for each $j \leqslant i$ we have a group morphism $\mu_{ij} : G_i \to G_j$ such that for all $i \in I$, $\mu_{ii} = \mathrm{Id}_{G_i}$ and $\mu_{ik} = \mu_{jk} \circ \mu_{ij}$ whenever $k \leqslant j \leqslant i$. Then (G_i, μ_{ij}) is a *directed system* and we can define the **direct limit** $\varinjlim_{i\in I} G_i$ as follows. Consider the direct sum $\bigoplus_i G_i$ of abelian groups, together with the subgroup R generated by elements of the form $x_i - \mu_{ij}(x_i)$ for all $j \leqslant i$ and $x_i \in G_i$. We define

$$\varinjlim_{i\in I} G_i := \bigoplus_i G_i/R,$$

that is, two elements in $g_i \in G$ and $g_j \in G$, $j \leqslant i$ are equivalent if and only if there is $k \in I$ such that $\mu_{ik}(g_i) = \mu_{ij}(g_i)$. If we let $\mu_i : G_i \to \varinjlim_i G_i$ be the restriction of the natural projection $\bigoplus_i G_i \to \varinjlim_i G_i$ restricted to G_i, then

(i) $\mu_i = \mu_j \circ \mu_{ij}$ whenever $i \leqslant j$;
(ii) every element in $\varinjlim_i G_i$ can be written as $\mu_j(x_j)$ for some $x_j \in M_j$;
(iii) the direct limit is characterised by the following *universal property*: Let G be a group with group morphisms $\alpha_i : G_i \to G$ such that $\alpha_i = \alpha_j \circ \mu_{ij}$ whenever $j \leqslant i$. Then there exists a unique group morphism $\alpha : \varinjlim_i G_i \to G$ such that $\alpha_i = \alpha \circ \mu$,

see for instance [AtMa, Exercice 2.14-16].

33. Remark. We can replace the indexing set I in the definition of the direct limit by any *cofinal* subset J. Recall that a subset J of a directed set I is cofinal if for all $i \in I$ there exists $j \in J$ with $j \leqslant i$. Here are two examples:

(i) We can define the stalk $\mathcal{F}_p$ of a complete presheaf by

$$\mathcal{F}_p = \varinjlim_{U\in\mathcal{U}(p)} \mathcal{F}_U,$$

where $\mathcal{U}(p)$ is a neighbourhood base of p. Then an element $\varphi \in \mathcal{F}_p$ is represented by a section $f \in \mathcal{F}(U)$ for a sufficiently small open set U containing p. If we denote by $\varphi = \mu_U(f) = [U, f]$ the resulting equivalence class, then $[U, f] = [V, g] \Leftrightarrow$ there exists $W \in \mathcal{U}(p)$, $W \subset U \cap V$ such that $\mu_{UW}(f) = f|_W = \mu_{VW}(g) = g|_V$. In particular, $[W, f|_W] = [V, g|_V]$.

(ii) Since the set of coverings given by bases of the topology of X is cofinal in the set of all coverings we can take the direct limit over bases of topology. A cohomology class in $H^q(X, \mathcal{F})$ is then represented by a cohomology class

$c_{\mathcal{U}} \in H^q(\mathcal{U}, \mathcal{F})$ by (ii) of 2.32 (which in turn is represented by a cocycle $\xi_{\mathcal{U}} \in Z^q(\mathcal{U}, \mathcal{F})$)

From Lemma 2.30 immediately follows

34. Corollary.

$$H^0(X, \mathcal{F}) \cong \Gamma(X, \mathcal{F})$$

Having defined cohomology for arbitrary sheaves we now face the problem to compute it for a given sheaf. In practice we will encounter three types of sheaves on a Riemann surface X: topological sheaves such as the constant sheaves $\underline{\mathbb{Z}}$ or $\underline{\mathbb{R}}$, "smooth" sheaves such as $\mathbb{C}^\infty_X$ or $\mathcal{A}^p_X$, and "holomorphic" sheaves such as $\mathcal{O}_X$ or Ω^p.

Topological sheaves. To compute cohomology groups like $H^q(X, \underline{\mathbb{Z}})$ one can appeal to results from algebraic topology. Assume that X admits the structure of a simplicial complex(this is always the case for a surface). Then simplicial cohomology $H^q(X, \mathbb{Z})$ is defined and we have

$$\check{H}^q(X, \underline{\mathbb{Z}}) \cong H^q(X, \mathbb{Z}).$$

Since simplicial cohomology equals ordinary singular cohomology we find, for instance for a compact Riemann surface X that

$$\check{H}^0(X, \underline{\mathbb{Z}}) = \mathbb{Z}, \quad \check{H}^0(X, \underline{\mathbb{Z}}) = \mathbb{Z}^{2g}, \quad \check{H}^0(X, \underline{\mathbb{Z}}) = \mathbb{Z},$$

where g denotes the genus of the Riemann surface, see Theorems C.7 and 9. Here, we first construct an isomorphism between the cohomology groups of a simplicial complex K_X underlying the topological space X, and the cohomology groups of an associated open covering $\mathcal{U}_K$. Towards this end recall that the *star of a vertex* $\mathrm{St}(\nu)$ of K_X is the interior of the union of all simplices containing ν as a vertex. Then $\mathcal{U}_K := \{\mathrm{St}(\nu) \mid \nu \text{ vertex of } K_X\}$ defines an open covering. Moreover, $\bigcap_{\alpha=0} 1\mathrm{St}(\nu_\alpha) \mid \varnothing$ and connected $\Leftrightarrow \nu_0, \ldots, \nu_q$ are the verteces of a q-simplex. We can then define a map $C^q(\mathcal{U}_K, \underline{\mathbb{Z}}) \to C^q(K_X, Z)$ (the latter group being the group of simplicial cochains) by sending f to $\sum f(\sigma)\sigma$ where the sum is being taken over the connected $\sigma \in N(\mathcal{U}_K)$. This induces an isomorphism $\check{H}^q(\mathcal{U}_K, \underline{\mathbb{Z}}) \cong H^q(K, \mathbb{Z})$ and taking succesive subdivisions of $K = K_X$ yields refinements $\mathcal{U}_K$ whch are cofinal in the set of open coverings. Passing to the limit gives $\check{H}^q(X, \underline{\mathbb{Z}}) \cong H^q(X, \mathbb{Z})$.

Smooth sheaves. Here, the so-called *long exact sequences* plays a key rôle. For a motivation, consider a short exact sequence of sheaves

$$0 \longrightarrow \mathcal{G} \xrightarrow{\iota} \mathcal{F} \xrightarrow{\pi} \mathcal{Q} \longrightarrow 0.$$

For any open set U we get an induced sequence at the level of the presheaves of sections,

$$0 \longrightarrow \mathcal{G}(U) \xrightarrow{\iota_*} \mathcal{F}(U) \xrightarrow{\pi_*} \mathcal{Q}(U).$$

As we have discussed above, π_* is not surjective in general. Cohomology can be regarded as a measure for the inexactness of this sequence (at least for $U = X$). For this we need to restrict our discussion to *paracompact (Hausdorff) spaces*, for instance surfaces. Recall that a Hausdorff space is called **paracompact** if every open covering has a locally finite refinement. In particular, it suffices to take the direct limit over locally finite coverings in the definition of cohomology.

35. Theorem (Long exact sequence) [Gu, Theorem 1]**.** *If X is a paracompact Hausdorff space, and if*

$$0 \longrightarrow \mathcal{G} \xrightarrow{\iota} \mathcal{F} \xrightarrow{\pi} \mathcal{Q} \longrightarrow 0$$

is an exact sequence of sheaves of abelian groups over X, then there exists a **long exact sequence**

$$0 \longrightarrow H^0(X,\mathcal{G}) \xrightarrow{\iota_*} H^0(X,\mathcal{F}) \xrightarrow{\pi_*} H^0(X,\mathcal{Q}) \xrightarrow{\delta_*} H^1(X,\mathcal{G}) \xrightarrow{\iota_*}$$

$$H^1(X,\mathcal{F}) \xrightarrow{\pi_*} H^1(X,\mathcal{Q}) \xrightarrow{\delta_*} H^2(X,\mathcal{G}) \longrightarrow \quad .$$

Here, ι_* and π_* are the induced maps on cohomology – they commute both with δ (so that they define maps between the cohomology groups with respect to $\mathcal{U}$) and with refinements (so that they induce maps at cohomology level $H^q(X)$).

Proof. We sketch the proof. For more details, see [Gu, Theorem 1].

Construction of the coboundary operator. First choose an open covering $\mathcal{U}$ of X. For each simplex $\sigma \in N(\mathcal{U})$ there is an induced exact sequence $0 \to \mathcal{G}(|\sigma|) \to \mathcal{F}(|\sigma|) \to \mathcal{Q}(|\sigma|)$. Since the coboundary groups $C^q(\mathcal{U},\mathcal{G})$ etc. are direct products of $\Gamma(|\sigma|,\mathcal{G})$ we get an induced exact sequence $0 \to C^q(\mathcal{U},\mathcal{G}) \stackrel{\iota_*}{\to} C^q(\mathcal{U},\mathcal{F}) \stackrel{\pi_*}{\to} C^q(\mathcal{U},\mathcal{Q})$. In order to obtain a short exact sequence we replace $C^q(\mathcal{U},\mathcal{Q})$ by the image $\pi_*(C^q(\mathcal{U},\mathcal{F})) =: \bar{C}^q(\mathcal{U},\mathcal{F})$. Then π_* and ι_* commute with the differential δ. If we define

$$\bar{H}^q(\mathcal{U},\mathcal{Q}) = \{f \in \bar{C}^q(\mathcal{U},\mathcal{Q}) \mid \delta f = 0\}/\delta\bar{C}^{q-1}(\mathcal{U},\mathcal{Q}),$$

we then get an induced exact sequence at cohomology level

$$H^q(\mathcal{U},\mathcal{G}) \xrightarrow{\iota_*} H^q(\mathcal{U},\mathcal{F}) \xrightarrow{\pi_*} \bar{H}^q(\mathcal{U},\mathcal{Q}).$$

We can now define $\delta_* : \bar{H}^q(\mathcal{U},\mathcal{Q}) \to H^{q+1}(\mathcal{U},\mathcal{G})$. Namely, take $[c] \in \bar{H}^q(\mathcal{U},\mathcal{Q})$. Then $c \in \bar{C}^q(\mathcal{U}\mathcal{Q})$ is in the image of π_*, i.e. there exists $f \in C^q(\mathcal{U},\mathcal{F})$ with $\pi_* f = t$. But $\delta\pi_* f = \pi_*\delta f = \delta c = 0$, so that there exists $g \in C^{q+1}(\mathcal{U},\mathcal{G})$ with $\iota_* g = f$. We let $\delta_*[c] = [g]$. Since ι_* is injective, we have indeed $\delta g = 0$, and the definition is independent of the choices made (check). Two issues remain: First, the independence of the covering and second, to get rid of $\bar{H}^q$.

Independence of the covering $\mathcal{U}$. Consider a refinement $\mu : \mathcal{V} \to \mathcal{U}$. We have two long cohomology sequences associated with $\mathcal{U}$ and $\mathcal{V}$ respectively, which are interrelated by μ^* (it is immediate to check that $\mu^*\bar{H}^q(\mathcal{U},\mathcal{F}) \subset \bar{H}^q(\mathcal{V},\mathcal{F})$). Since μ^* commutes with ι_*, π_* and δ_* taking the direct limit commutes with the cohomology squence and yields an exact cohomology sequence

$$\ldots \longrightarrow H^q(X,\mathcal{G}) \xrightarrow{\iota_*} H^q(X,\mathcal{F}) \xrightarrow{\pi_*} H^q(X,\mathcal{Q}) \xrightarrow{\delta_*} H^{q+1}(X,\mathcal{G}) \longrightarrow \ldots.$$

$\bar{H}^q(X,\mathcal{Q}) = H^q(X,\mathcal{Q})$. Here we use the paracompactness of the space. We will show that for a given cochain $c \in C^q(\mathcal{U},\mathcal{Q})$ there exists a refinement $\mu : \mathcal{V} \to \mathcal{U}$ and $f \in C^q(\mathcal{V},\mathcal{F})$ such that $\mu^* c = \pi_* f$, that is, any $\mathcal{Q}$-cochain lies in the image of π_*, possibly after refining the covering. Since X is paracompact we may assume that $\mathcal{U}$ is locally finite. For each $p \in X$ we then choose an open neighbourhood V_p of p in X such that

(i) if $V_p \cap U_\alpha \neq \varnothing \Rightarrow V_p \subset U_\alpha$: Since $\mathcal{U}$ is locally finite, there exists an open neighbourhood V_p of p which intersects only finitely many U_{α_i}. Shrinking V_p further if necessary implies that $V_p \subset \bigcap_i U_{\alpha_i}$.

(ii) If $\sigma = (U_0, \ldots, U_q) \in N(\mathcal{U})$, $h \in \Gamma(|\sigma|, \mathcal{Q})$ and $V_p \subset |\sigma| \Rightarrow h|_{V_p} = \pi_*(g)$ for some $g \in \Gamma(V_p, \mathcal{F})$: Use the fact that $\mathcal{Q}$ is a quotient sheaf, shrinking V_p further if necessary.

For each $p \in X$ define a refinement $\mathcal{V} = \{V_p\}_{p \in X}$ by choosing $U_p = \mu(V_p) \in \mathcal{U}$ which is possible by (i). For any q-simplex $\tau = (V_{p_0}, \ldots, V_{p_q}) \in N(\mathcal{V})$ we have $|\tau| \subset U_{p_0} \cap \ldots \cap U_{p_q}$. Then $V_{p_0} \cap U_{p_i} \neq \varnothing$ so that $V_{p_0} \subset U_{p_i}$ again by (i). It follows that $|\tau| \subset V_{p_0} \subset U_{p_0} \cap \ldots \cap U_{p_q} = |\mu(\tau)|$. Hence if $c \in C^q(\mathcal{U}, \mathcal{Q})$, then

$$\mu^* c(\tau) = c(\mu(\tau))|_{|\tau|} = \big(c(\mu(\tau))|_{V_{p_0}}\big)|_{|\tau|}.$$

However, $c(\mu(\tau))|_{V_{p_0}} = \pi_* f$ for a section $f \in \Gamma(V_{p_0}, \mathcal{F})$ by (iii) with $\sigma = \mu(\tau)$. This proves the claim and finishes the proof. □

In view of the long exact sequence, the following property of sheaves is of interest.

36. Definition (acylic sheaf). A sheaf $\mathcal{F}$ is called *acyclic* $\Leftrightarrow H^q(X, \mathcal{F}) = 0$ for $q > 0$.

37. Examples of acyclic sheaves [Go]**.**

(i) **soft** (“mou”) sheaves: If $A \subset X$ is a closed subset, then $\Gamma(X, \mathcal{F}) \to \Gamma(A, \mathcal{F})$ is surjective.
(ii) **flabby** (“flasque”) sheaves: If $U \subset X$ is an open subset, then $\Gamma(X, \mathcal{F}) \to \Gamma(U, \mathcal{F})$ surjective.
(iii) fine sheaves: We will treat these below.

This gives rise to the following way of computing sheaf cohomology in general provided one has an **acyclic resolution** of a sheaf $\mathcal{F}$, i.e. an exact sequence of the form

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}_0 \xrightarrow{d_0} \mathcal{F}_1 \xrightarrow{d_1} \mathcal{F}_2 \xrightarrow{d_2} \ldots$$

where the sheaves $\mathcal{F}_i$ are acyclic sheaves, and $d_i : \mathcal{F}_i \to \mathcal{F}_{i+1}$ are sheaf morphisms.

38. Theorem [Gu, Theorem 3]**.**) *If X is paracompact, and $\mathcal{F}$ is a sheaf admitting an acyclic resolution, then*

$$H^q(X, \mathcal{F}) \cong \ker d_q(X)/\operatorname{im} d_{q-1}(X)$$

for $q > 0$.

Proof. Let $\mathcal{K}_i = \ker d_i \subset \mathcal{F}_i$. First we get a short exact sequence of the form

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}_0 \xrightarrow{d_i} \mathcal{K}_1 \longrightarrow 0$$

from which we deduce that $H^1(\mathcal{F}) \cong H^0(\mathcal{K}_1)/\operatorname{im} d_0$ and $H^{q+1}(\mathcal{F}) \cong H^q(\mathcal{K}_1)$ for $q \geqslant 1$ since $\mathcal{F}_0$ is fine. Now the exact sequences

$$0 \longrightarrow \mathcal{K}_i \longrightarrow \mathcal{F}_i \xrightarrow{d_i} \mathcal{K}_{i+1} \longrightarrow 0$$

yield $H^q(\mathcal{K}_1) \cong H^{q-1}(\mathcal{K}_2) \cong \ldots \cong H^1(\mathcal{K}_q) \cong H^0(\mathcal{K}_{q+1})/\operatorname{im} d_q$. □

39. Example: De Rham cohomology. Though one can show that an acyclic resolution always exists. However, in practice one uses concrete natural resolutions. For instance, we will see in a moment, at least in the case of the differentiable structure underlying a Riemann surface, that the sheaves of differential forms $\mathcal{A}^p_X$ are fine. In particular, for any differentiable manifold M we get a fine resolution

$$0 \longrightarrow \underline{\mathbb{C}} \longrightarrow \mathcal{A}^0_M \xrightarrow{d_0} \mathcal{A}^1_M \xrightarrow{d_1} \dots$$

In particular, $\check{H}^q(M, \underline{\mathbb{C}}) \cong \ker d_q / \operatorname{im} d_{q-1} =: H^q_{DR}(M)$, where the right hand side defines the so-called **de Rham cohomology**. By the considerations above we obtain in particular that $H^q(M, \mathbb{C}) \cong H^q_{DR}$ (de Rham's theorem) – although de Rham cohomology is defined in terms of the differentiable structure, the resulting cohomology theory is a topological invariant!

We say that a family $\{\eta_\alpha\}_{\alpha\in\Lambda}$ of sheaf morphisms $\mathcal{F} \to \mathcal{F}$ is a **partition of unity for the sheaf** $\mathcal{F}$ subordinate to the locally finite covering $\mathcal{U} = \{U_\alpha\}_{\alpha\in\Lambda}$ if

- $\eta_\alpha(\mathcal{F}_p) = 0$ for all $p \notin U_\alpha$;
- $\sum_\alpha \eta_\alpha(g) = g$ for all $g \in \mathcal{F}$.

As for usual partitions of unity, local finiteness ensures that the sum $\sum_a \eta_a(g)$ is finite. We say that $\mathcal{F}$ is a **fine** sheaf if it admits a partition of unita for any locally finite covering. For instance, the sheaf of smooth functions C^∞_X over a Riemann surface is fine, for a usual partition of functions $\{(U_\alpha, f_\alpha)\}$ (as we used to define integration) induces a partition of unity by extending f_α by 0 to all of x which acts on C^∞ by multiplication.

40. Theorem (fine sheaves are acyclic) [Gu, Theorem 2]**.** *If $\mathcal{U} = \{U_\alpha\}$ is a locally finite covering and $\{\eta_\alpha\}$ a partition of unity of $\mathcal{F}$ subordinate to $\mathcal{U} \Rightarrow H^q(\mathcal{U}, \mathcal{F}) = 0$ for all $q > 0$. In particular, $H^q(X, \mathcal{F}) = 0$ for all $q > 0$ if X is paracompact and $\mathcal{F}$ fine.*

Proof. Let $f \in Z^q(\mathcal{U}, \mathcal{F})$, $q > 0$. We want to show that f is *exact*, that is, $f = \delta g$ for some g. The η_α induce morphisms $\eta_{\alpha*}$ on $C^q(\mathcal{U}, \mathcal{F})$ such that $f = \sum \eta_{\alpha*} f$, so we need only to show that $\eta_{\alpha*} f = \delta g_\alpha$ for any fixed α. We define g_α as follows. Let $\tau = (V_0, \dots, V_{q-1})$ be a $q-1$-simplex. If $U_\alpha \cap |\tau| = \varnothing$, then $g_\alpha(\tau) = 0 \in \Gamma(|\tau|, \mathcal{F})$. Otherwise, let

$$g_\alpha(\tau) = e_{|\tau|}(\eta_{\alpha*} f)(U_\alpha, \tau)$$

where $e_{|\tau|} : \Gamma(U_\alpha \cap |\tau|) \to \Gamma(|\tau|)$ denotes extension by zero. Then $g \in C^{q-1}(\mathcal{U}, \mathcal{F})$ and if $\sigma = (U_0, \dots, U_q)$, $\tau_i = (U_0, \dots, U_{i-1}, U_{i+1}, \dots, U_q)$, we have by a short computation

$$\delta g_\alpha(\sigma) = \sum_{i=0}^q (-1)^i g_\alpha(\sigma_i)|_{|\sigma|} = \eta_{\alpha*} f(\sigma) - e_{|\sigma|} \delta \eta_{\alpha*} f = \eta_{\alpha*} f(\sigma)$$

for $\delta f = 0$. Defining $g = \sum_\alpha g_\alpha$ yields $f(\sigma) = \sum_\alpha \eta_{\alpha*} f(\sigma) = \sum_\alpha (\delta g_\alpha)(\sigma) = (\delta g)$ as required. □

A further application of fine sheaves is

41. Theorem (Leray) [Gu, Theorem 5]**.** *Let $\mathcal{F}$ be a sheaf of abelian groups over a paracompact space X, and let $\mathcal{U} = \{U_\alpha\}$ be a* **Leray covering***, i.e. an open covering of X such that $H^q(|\sigma|, \mathcal{F}) = 0$ for all $\sigma \in N(\mathcal{U})$ and $q \geqslant 1$. Then*

$$H^q(X, \mathcal{F}) \cong H^q(\mathcal{U}, \mathcal{F})$$

for all $q \geqslant 0$.

Proof. Pick an acyclic resolution

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}_0 \xrightarrow{d_0} \mathcal{F}_1 \xrightarrow{d_1} \mathcal{F}_2 \xrightarrow{d_2} \dots$$

for $\mathcal{F}$ (as remarked above, this always exists). It follows that $H^q(X,\mathcal{F})$ is isomorphic to $\ker d_q(X)/\operatorname{im} d_{q-1}(X)$. Restriction to $|\sigma|$ for $\sigma \in N(\mathcal{U})$ yields an acyclic resolution which computes $H^q(|\sigma|,\mathcal{F})$. Since $H^q(|\sigma|,\mathcal{F}) = 0$ it follows that

$$0 \longrightarrow \Gamma(|\sigma|,\mathcal{F}) \longrightarrow \Gamma(|\sigma|,\mathcal{F}_0) \xrightarrow{d_0*} \Gamma(|\sigma|,\mathcal{F}_1) \xrightarrow{d_1*} \Gamma(|\sigma|,\mathcal{F}_2) \xrightarrow{d_2*} \dots$$

defines actually an exact sequence. As the cochain groups are just direct products of groups $\Gamma(|\sigma|,\mathcal{F}_i)$ it follows that we also obtain an exact sequence

$$0 \longrightarrow C^q(\mathcal{U},\mathcal{F}) \longrightarrow C^q(\mathcal{U},\mathcal{F}_0) \xrightarrow{d_0*} C^q(\mathcal{U},\mathcal{F}_1) \xrightarrow{d_1*} C^q(\mathcal{U},\mathcal{F}_2) \xrightarrow{d_2*} \dots.$$

The morphisms commute with the differential. Consequently, these sequences can be grouped together to give the following big commutative diagramm:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Gamma(X,\mathcal{F}) \arrow[r] \arrow[d] & \Gamma(X,\mathcal{F}_0) \arrow[r, "d_0"] \arrow[d] & \Gamma(X,\mathcal{F}_1) \arrow[r, "d_1"] \arrow[d] & \Gamma(X,\mathcal{F}_2) \arrow[r, "d_2"] \arrow[d] & \dots \\
0 \arrow[r] & C^0(\mathcal{U},\mathcal{F}) \arrow[r] \arrow[d, "\delta"] & C^0(\mathcal{U},\mathcal{F}_0) \arrow[r, "d_0*"] \arrow[d, "\delta"] & C^0(\mathcal{U},\mathcal{F}_1) \arrow[r, "d_1*"] \arrow[d, "\delta"] & C^0(\mathcal{U},\mathcal{F}_2) \arrow[r, "d_2*"] \arrow[d, "\delta"] & \dots \\
0 \arrow[r] & C^1(\mathcal{U},\mathcal{F}) \arrow[r] \arrow[d, "\delta"] & C^1(\mathcal{U},\mathcal{F}_0) \arrow[r, "d_0*"] \arrow[d, "\delta"] & C^1(\mathcal{U},\mathcal{F}_1) \arrow[r, "d_1*"] \arrow[d, "\delta"] & C^1(\mathcal{U},\mathcal{F}_2) \arrow[r, "d_2*"] \arrow[d, "\delta"] & \dots \\
0 \arrow[r] & C^2(\mathcal{U},\mathcal{F}) \arrow[r] \arrow[d, "\delta"] & C^2(\mathcal{U},\mathcal{F}_0) \arrow[r, "d_0*"] \arrow[d, "\delta"] & C^2(\mathcal{U},\mathcal{F}_1) \arrow[r, "d_1*"] \arrow[d, "\delta"] & C^2(\mathcal{U},\mathcal{F}_2) \arrow[r, "d_2*"] \arrow[d, "\delta"] & {} \\
 & \vdots & \vdots & \vdots & \vdots &
\end{tikzcd}$$

It follows from our initial considerations that all the rows except for the first are exact. Similarly, all the columns except for the first are exact. Since the inexactness of these sequences is precisely measures by cohomology, a simple diagramm chase gives the result. □

Though the existence of a Leray covering is a priori unclear we will see powerful applications of this theorem later on.

Holomorphic sheaves. These are at the heart of theory of Riemann surfaces and need to be computed individually.

42. Definition (Dolbeault cohomology). Let X be a (compact) Riemann surface, and let $\mathcal{O} = \mathcal{O}_X$ be the sheaf of holomorphic functions. Then $H^q(X,\mathcal{O})$ is called the **q-th Dolbeault cohomology group**.

We know already that $H^0(X,\mathcal{O}) = \mathbb{C}$, that is, the only globally defined holomorphic functions are the constant functions. To compute $H^q(X,\mathcal{O})$ (at leasfor $q > 0$, let

$U \subset \mathbb{C}$ be an open set. Recall that the Cauchy-Riemann operator $\bar{\partial} = (\partial_x + i\partial_y)/2$ which induces the sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow C^\infty_X \xrightarrow{\bar{\partial}} C^\infty_X$$

by acting on germs represented by holomorphic functions defined over charts. We then prove the

43. Theorem [Gu, Theorem 4]**.** *Let $U \subset \mathbb{C}$ be a domain, and let $g \in C^\infty(U) \Rightarrow$ There exists $f \in C^\infty(U)$ such that $\bar{\partial} f = g$.*

Proof. We proceed in two steps

Step 1. *Let $D \subset U$ be a domain such that $\bar{D}$ is compact and contained in $U \Rightarrow$ There exists $f \in C^\infty(U)$ such that $\bar{\partial} f|_D = g|_D$.* Pick a smooth functoin ρ on $\mathbb{C}$ such that $\rho(z) = 1$ for $z \in \bar{D}$ and $\rho(z) = 0$ for $z \in \mathbb{C}\backslash U$, and $\operatorname{supp}\rho$ is compact. Then we can define a smooth function h by

$$h(z) = \rho(z)g(z) \text{ for } z \in U, \quad h(z) = 0 \text{ for } z \in \mathbb{C}\backslash U.$$

Note that $h|_D = g|_D$. We put

$$f(z) = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{h(z+\xi)}{\xi} d\xi \wedge d\bar{\xi}.$$

The integral is well-defined for we have, passing to polar coordinates $\xi = re^{i\theta}$, that $(d\xi \wedge d\bar{\xi})/\xi = -2ie^{-i\theta}dr \wedge d\theta$ so that we are integrating a smooth function with compact support. In particular, differentiation commutes with integration. Note that by the chain rule differentiation of $h(z+\xi)$ is symmetric with respect $\bar{\xi}$ and $\bar{z}$. Moreover, $\partial_{\bar{\xi}}$ is ξ-linear. It follows that

$$\begin{aligned}
\partial_{\bar{z}} f(z) &= \frac{1}{2\pi i}\int_{\mathbb{C}} \partial_{\bar{z}} h(z+\xi)\frac{d\xi \wedge d\bar{\xi}}{\xi} \\
&= \frac{1}{2\pi i}\int_{\mathbb{C}} \partial_{\bar{\xi}} h(z+\xi)\frac{d\xi \wedge d\bar{\xi}}{\xi} \\
&= \frac{1}{2\pi i}\int_{\mathbb{C}} \partial_{\bar{z}} \frac{h(z+\xi)}{\xi} d\xi \wedge d\bar{\xi}.
\end{aligned}$$

Next fix $z \in \mathbb{C}$, a disc D_R whose radius R is large enough that $\operatorname{supp} h \subset D_R$, and a disc D_ϵ such that $\bar{D}_\epsilon \subset D_R$. We denote by γ_R respectively γ_ϵ the circles bounding D_R and D_ϵ with positive orientation. Then

$$\begin{aligned}
2\pi i \partial_{\bar{z}} f(z) &= \lim_{\epsilon \to 0} \int_{D_R \backslash D_\epsilon} \partial_{\bar{\xi}} \Big(\frac{h(z+\xi)}{\xi}\Big) d\xi \wedge d\bar{\xi} \\
&= \lim_{\epsilon \to 0} \int_{D_R \backslash D_\epsilon} d\Big(\frac{h(z+\xi)}{\xi} d\xi\Big) d\bar{\xi},
\end{aligned}$$

and Stokes' Theorem (8) implies that

$$2\pi i \partial_{\bar{z}} f = \lim_{\epsilon \to 0} \int_{\gamma_\epsilon} \frac{h(z+\xi)}{\xi} d\xi.$$

Parametrising γ_ϵ by $\xi = \epsilon e^{i\theta}$ yields

$$\begin{aligned}
2\pi i \partial_{\bar{z}} f(z) &= \lim_{\epsilon \to 0} \int_{\theta=0}^{2\pi} h(z+\epsilon e^{i\theta}) i d\theta \\
&= \int_{\theta=0}^{2\pi} h(z) i d\theta \\
&= 2\pi i h(z).
\end{aligned}$$

Hence f is the desired function.

Step 2. *Conclusion.* Select a sequence of domains $D_n \subset U$ as in the first step such that

- $\bar{D}_n \subset D_{n+1}$;
- $U = \bigcup_{n=1}^{\infty} D_n$;
- any $f \in \mathcal{O}(D_{n-1})$ can be approximated uniformly well by $f_i \in \mathcal{O}(D_n)$ over $\bar{D}_{n-2}$.

The existence of such a sequence is garantued by Runge's approximation theorem, see [Fo, Section 25, in particular Theorem 25.4]. We claim that there exists a sequence of functions $f_n \in C^\infty(D_n)$ such that $\bar{\partial} f = g$ over D_n and $|f_n(z) - f_{n-1}(z)| < 2^{-n}$ for all $z \in \bar{D}_{n-2}$. We proceed by induction. By the first step there exists a smooth function $h_n \in C^\infty(U)$ such that $\bar{\partial} h_h = g$ over D_n. For $n = 0$ and $n = 1$ there is nothing more to show if we put $f_i = h_i$. If $n \geqslant 2$, then h_n and f_{n-1} are both smooth over D_{n-1}, and $\bar{\partial}(h_n - f_{n-1}) = 0$ over D_{n-1}, that is, $h_n - f_{n-1} \in \mathcal{O}(D_{n-1})$. Now we choose an aproximating function $h \in \mathcal{O}(D_n)$ such that $\sup_{z \in \bar{D}_{n-2}} |h_n(z) - f_{n-1}(z) - h(z)| < 2^{-n}$ and put $f_n = h_n - h$.

The resulting sequence $\{f_n(z)\}$ is Cauchy and therefore converges to some limiting value $f(z)$. Indeed, consider

$$f(z) = f_{n+2}(z) + \sum_{m=n+2}^{\infty} (f_{m+1}(z) - f_m(z)).$$

Since $|f_{m+1}(z) - f_m(z)| < 2^{-m}$ for $z \in D_n \subset D_{m-2}$ the series f_n converges with respect to the supremum norm over D_n. Since the individual terms in the series are holomorphic, so is their sum. Consequently, $f \in C^\infty(D_{n-2})$ and $\bar{\partial} f = g$. Since the fmaily D_n exhausts U, the result follows.

□

For a general Riemann surface X Theorem 2.43 implies that we have an exact sequence (the **Dolbeault sequence of** X)

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{A}_X^{0,0} \xrightarrow{\bar{\partial}} \mathcal{A}_X^{0,1} \longrightarrow 0 \tag{9}$$

Since the $\mathcal{A}_X^{p,q}$ are fine sheaves we immediately obtain the

44. Corollary (Dolbeault). *$H^q(X, \mathcal{O}) = 0$ for $q > 1$.*

45. Remark. We will see later that $H^1(X, \mathcal{O}) \cong \mathbb{C}^g$, where g is the genus of the underlying surface.

2.2. **Line bundles and divisors.** We have seen that on a compact Riemann surface only constants define global holomorphic functions. Now considering $\mathcal{O}_X$ as a sheaf it is *free* in the sense that $\mathcal{O}_X(X) = \Gamma(X, \mathcal{O}_X)$ are just functions $X \to \mathbb{C}$, that is, a global section $\sigma : X \to \mathcal{O}_X$ can be represented by global holomorphic functions $f : X \to \mathbb{C}$, $\sigma(p) = (p, f(p))$. In this section we will introduce *line bundles*. Morally, these are sheaves which are locally of the form $U \times \mathbb{C}$. Unlike $\mathcal{O}_X$ these sheaves *can* possess nontrivial global sections. Closely related to this are *divisors* which we will investigate first.

Divisors. Recall that $\mathcal{O}_X^*$ is the germ of nowhere vanishing holomorphic functions, that is $f \in \mathcal{O}_X^*(U) \Leftrightarrow f \in \mathcal{O}_X(U)$ and $f(z) \neq 0$ for all $z \in U$. Furthermore, $\mathcal{M}^*$ is the sheaf of not identically vanishing meromorphic functions, that is, $f \in \mathcal{M}_X^*(U) \Leftrightarrow f \not\equiv 0$.

46. Definition (sheaf of divisors). The quotient sheaf $\mathcal{D}_X = \mathcal{M}_X^*/\mathcal{O}_X^*$ is the **sheaf of germs of divisors**. A section $D \in \Gamma(U, \mathcal{D}_X)$ will be called a divisor on U. (Note that these definitions make also sense on noncompact RIemann surfaces).

A germ $D_p \in \mathcal{D}_p$, $p \in X$, is thus an equivalence class of a nontrivial meromorphic function which is defined up to an invertible holomorphic function. This leads to a particularly simple description of divisors in terms of the order function o_a 1.53. Clearly, o_a descends to the quotient $\mathcal{D}_p = \mathcal{M}_{X,p}^*/\mathcal{O}_{X*,p}$ and actually induces an isomorphism $\mathcal{D}_p \cong \mathbb{Z}$. Furthermore, a section $D \in \Gamma(U, \mathcal{D})$ is locally represented by a nowhere vanishing meromorphic function whose zeroes and pôles are isolated. Applying this isomorphism pointwise implies that $o_p(D(p)) = 0$ except for a discrete subset of U which is finite if U is relatively compact in X. Consequently, we can identify D with the locally finite formal sum $\sum_{p \in U} o_p(D(p))p$ of poitns with coefficients in $\mathbb{Z}$. The multiplicative structure of $\mathcal{D}$ corresponds to an additive structure on these formal sums for $o_p(f \cdot g) = o_p(f) + o_p(g)$. We therefore get an alternatve description of $\mathcal{D}$, namely as the complete presheaf $\mathcal{D}(U)$ consisting of sections $U \to \mathbb{Z}$ which are zero except for a discrete subset of U together with the natural restriction maps. Moreover, this obviously defines a *flasque* sheaf so that in particular, $H^q(X, \mathcal{D}) = 0$ for $q \geqslant 1$. Note that there is a naturl partial ordering for divisors. We say that $D \in \mathcal{D}_X(U)$ is **positive** if $D = \sum a_p p$ with integers $a_p \geqslant 0$. We write $D \geqslant 0$ and $D \geqslant D' \Leftrightarrow D - D' \geqslant 0$.

47. Example: the divisor of a meromorphic function. With any nontrivial meromorphic function $f \in \mathcal{M}_X^*(U)$ we can associate a divisor $(f) := \sum o_p(f_p)p$. Divisors of this form are called **principal**. Note that a prinicpal divisor is positive $\Leftrightarrow f$ is holomorphic.

The map which associates a principal divisor with a meromorphic function gives rise to the exact sequence

$$0 \longrightarrow \mathcal{O}_X^* \xrightarrow{\iota} \mathcal{M}_X^* \xrightarrow{(\cdot)} \mathcal{D}_X \longrightarrow 0,$$

where ι is the natural inclusion mapping. In particular, we get an induced map $\Gamma(X, \mathcal{M}_X^*) \xrightarrow{(\cdot)} \Gamma(X, \mathcal{D}_X)$. Since $\mathcal{D}_X$ is flasque, globally defined divisors exist in abundance. The existence of nontrivial meromophic functions is less obvious. A first existence result is this:

48. Theorem (Weierstrass) [Gu, Theorem 6]**.** *If U is a domain in $\mathbb{C}$, then we have an exact sequence of groups*

$$0 \longrightarrow \Gamma(U, \mathcal{O}_U^*) \xrightarrow{\iota*} \Gamma(U, \mathcal{M}_U^*) \xrightarrow{(\cdot)*} \Gamma(U, \mathcal{D}_U) \longrightarrow 0.$$

In particular, any divisor on U is principal.

Proof. Surjectivity follows from the ling exact sequence provided we can show that $H^1(U, \mathcal{O}_U^*) = 0$. Let $X = U$. From the exact sequence

$$0 \longrightarrow \underline{\mathbb{Z}} \xrightarrow{\iota} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \longrightarrow 0$$

we deduce the exact sequence

$$H^1(X, \mathcal{O}_X) \longrightarrow H^1(X, \mathcal{O}_X^*) \longrightarrow H^2(X, \mathbb{Z}) \longrightarrow H^2(X, \mathcal{O}_X).$$

But we know already that $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$ from which we obtain $H^1(X, \mathcal{O}^*) \cong H^2(X, \underline{\mathbb{Z}})$. But since X is noncompact, $H^2(X, \underline{\mathbb{Z}}) = 0$ (this is a general topological fact for noncompact surfaces) from which the result follows. □

49. Remark. More generally, this result holds for any noncompact Riemann surface X as they satisfy $H^{1,2}(X, \mathcal{O}_X) = 0$, see [Forster]. One can actually explicitely construct the meromorphic function via Weierstrass' factor theorem A.28. The virtue of our approach lies in the relative simplicity of the proof, once the vanishing of $H^{1,2}(X, \mathcal{O}_X)$ is established.

50. Corollary. *If U is a domain in $\mathbb{C}$, then* $\operatorname{Quot} \mathcal{O}(U) = \mathcal{M}(U)$.

Proof. It is clear that $\operatorname{Quot} \mathcal{O}(U) \subset \mathcal{M}(U)$. Conversely, let $f \in \mathcal{M}(U)$. By the Weierstrass Theorem 2.48 we can find a holomorphic function $h \in \mathcal{O}(U)$ whose divisor gives precisely the pole divisor of f (i.e. the divisor consisting of the poles of f together with their multiplicity). In particular, $g := h \cdot f \in \mathcal{O}(U)$ so that $f = g/h \in \operatorname{Quot} \mathcal{O}(U)$. □

Next we investigate the exact sequence $0 \to \mathcal{O}^* \to \mathcal{M}^* \to \mathcal{D} \to 0$ using the fact that $\mathcal{D}$ is an acyclic sheaf.

51. Definition (divisor class group). Two divisors D_1 and D_2 in $\Gamma(X, \mathcal{D})$ are **linearly equivalent** if $D_1 - D_2 = (f)$ for $f \in \Gamma(X, \mathcal{M}^*)$. The group

$$\mathrm{Cl}\,(X) = \Gamma(X, \mathcal{D})/(\Gamma(X, \mathcal{M}^*))$$

is called the **divisor class group** of X.

The long exact sequence yields the short exact sequence

$$0 \longrightarrow \mathrm{Cl}\,(X) \longrightarrow H^1(X, \mathcal{O}^*) \longrightarrow H^1(X, \mathcal{M}^*) \longrightarrow 0$$

We investigate the cohomology group $H^1(X, \mathcal{O}^*)$ next.

Holomorphic line bundles. We start with the

52. Definition (holomorphic line bundles). A **holomorphic line bundle** ξ is an element in $H^1(X, \mathcal{O}^*)$. We call $H^1(X, \mathcal{O}^*)$ the **group of holomorphic line bundles**.

As explained in 2.32 and the subsequent remark, we can represent a holomorphic line bundle ξ by a cocycle which we write $\{\xi_{\alpha\beta}\} \in \mathcal{Z}^1(\mathcal{U}, \mathcal{O}^*)$ for a base $\mathcal{U} = \{U_\alpha\}$ of the topology of X. Writing the group operation multiplicatively the cocycle condition implies

$$\xi_{\alpha\beta} \cdot \xi_{\beta\gamma}|_{U_{\alpha\beta\gamma}} = \xi_{\alpha\gamma}|_{U_{\alpha\beta\gamma}},$$

where $U_{\alpha\beta} = U_\alpha \cap U_\beta$, $U_{\alpha\beta\gamma} = U_\alpha \cap U_\beta \cap U_\gamma$ etc. and where $U_{\alpha\beta\gamma} \neq \varnothing$. It follows that if $V = U_{\alpha\beta} \neq \varnothing$ and $V = \bigcup_{V \supset U_\gamma \in \mathcal{U}}$ is a covering by open sets in $\mathcal{U}$ ($\mathcal{U}$ is a base of the topology!), then

$$\xi_{\alpha\beta} = \xi_{\beta\alpha}^{-1}.$$

Indeed, $1 = (\delta\xi)_{\alpha\beta\gamma} = \xi_{\beta\gamma}\xi_{\alpha\gamma}^{-1}\xi_{\alpha\beta}|_{U_\gamma} \in \mathcal{O}^*(U_\gamma)$ and $1 = (\delta\xi)_{\beta\alpha\gamma} = \xi_{\alpha\gamma}\xi_{\beta\gamma}^{-1}\xi_{\beta\alpha}|_{U_\gamma} \in \mathcal{O}^*(U_\gamma)$. Hence $1 = (\delta\xi)_{\beta\alpha\gamma} \cdot (\delta\xi)_{\alpha\beta\gamma} = (\xi_{\alpha\beta} \cdot \xi_{\beta\alpha})|_{U_\gamma}$.

Next we associate a sheaf $\mathcal{O}_\xi$ with every $\xi \in H^1(X, \mathcal{O}^*)$. For each α, we let $\mathcal{O}_\xi(U_\alpha) = \Gamma(U_\alpha, \mathcal{O}) = \mathcal{O}_\xi(U_\alpha)$. However, for $U_\beta \subset U_\alpha$ we let

$$\rho_{\alpha\beta}(f)(p) = \xi_{\beta\alpha}(p) \cdot f(p)$$

instead of the usual restriction morphisms. The cocycle condition (2.2) garantuees that $\rho_{\beta\gamma} \circ \rho_{\alpha\beta} = \rho_{\alpha\gamma}$. In particular, $\mathcal{O}_\xi(U_\alpha)$ together with the "restriction functions" $\rho_{\alpha\beta}$ determine a presheaf since they are defined for a base of the topology, cf. Remark2.26. It is easy to see that it is complete and independent of the representing cocycle.

53. Definition (sheaf of holomorphic cross-sections). We call the sheaf $\mathcal{O}_\xi$ just constructed the **sheaf of holomorphic cross-sections of** ξ. We call the cocycle $\xi_{\alpha\beta}$ representing ξ an **atlas of** ξ if the underlying base of topology consists of coordinate neighbourhoods of X and $\mathcal{O}_\xi(U_\alpha) \cong \mathcal{O}(U_\alpha)$.

In particular, a global section $s \in \Gamma(X, \mathcal{O}_\xi)$ is determined by a collection of locally defined holomorphic functions $s_\alpha = \rho_{XU_\alpha}(s) \in \mathcal{O}(U_\alpha)$ which satisfy $s_\alpha|_{U_{\alpha\beta}} = \xi_{\alpha\beta} s_\beta|_{U_{\alpha\beta}}$ whenever $U_{\alpha\beta} \neq \varnothing$. Indeed, let $p \in U_{\alpha\beta}$ and let $U_{\alpha\beta} \supset U_\gamma$ be an open set in $\mathcal{U}$ containing p. By design, $\rho_{\alpha\gamma}(s_\alpha)(p) = \rho_{XU_\gamma}(s)(p) = \rho_{\beta\gamma}(s_\beta)(p)$ whence $\xi_{\gamma\alpha}(p) \cdot s_\alpha(p) = \xi_{\beta\gamma}(p) \cdot \xi_{\beta\alpha}(p) \cdot s_\alpha(p) = \xi_{\beta\gamma}(p) \cdot s_\beta(p)$.

54. Remark. To understand the terminology, we interpret $\xi = \{\xi_{\alpha\beta}\}$ given by an atlas geometrically by considering the cocycle as a family of holomorphic functions $\xi_{\alpha\beta} : U_{\alpha\beta} \to \mathbb{C}^* = \mathrm{GL}(1, \mathbb{C})$. We can then define

$$L_\xi = \bigsqcup U_\alpha \times \mathbb{C} / \sim_\xi,$$

where two elements $(z, s) \in U_\alpha \times \mathbb{C}$ and $(w, t) \in U_\beta \times \mathbb{C}$ are equivalent if and only if $w = z$ and $s = \xi_{\alpha\beta} t$. Clearly, this is an equivalence relation. The natural map $\bigsqcup U_\alpha \times \mathbb{C} \to L_\xi$ topologises this space. Furthermore, we have a natural projection $\pi_\xi : L_\xi \to X$ whose fibre is just $\mathbb{C}$. In a sense, it is like a sheaf except that its topology is of product type, that is, locally it is homeomorphic to $U_\alpha \times \mathbb{C}$. L_ξ is calle a **holomorphic line bundle**. In particular, it is an example of a higher dimensional complex manifold. As for sheave we can consider sections $\sigma : U \to L_\xi$ which satisfy $\pi_\xi \circ \sigma = \mathrm{Id}_U$. Since L is itself a complex manifold we can require these to be holomorphic. The holomorphic sections over U are then just given by $\mathcal{O}_\xi$.

Of course, L_ξ depends a priori on the cocycle $\xi_{\alpha\beta}$ rather than the cohomology class ξ. However, there is a natural notion of a morphism of a line bundle, and one can show that cohomologous cocycles give rise to holomorphic line bundles. Summarising, a cohomology class of ξ determines an isomorphism class of line bundles L_ξ.

55. Example. The trivial cohomology class $\mathbf{1}$ gives rise to the **holomorphically trivial line bundle** $X \times \mathbb{C}$, i.e. $\mathcal{O}_\xi = \mathcal{O}_X$. In particular, any global section must be constant if X is compact, i.e. $H^0(X, \mathcal{O}_{\mathbf{1}}) = H^0(X, \mathcal{O}_X) \cong \mathbb{C}$.

In fact, we have the

56. Lemma. *A line bundle ξ is holomorphic trivial, i.e. $\xi = 1 \in H^1(X, \mathcal{O}^*)$ $\Leftrightarrow$ there exists a global section $s \in \Gamma(X, \mathcal{O}_\xi^*)$, that is, s is nowhere vanishing.*

Proof. $\Rightarrow$) Then $\Gamma(X, \mathcal{O}_\xi) = \Gamma(X, \mathcal{O})$ so that any nontrivial constant will do.

$\Leftarrow$) Assume that $s \in \Gamma(X, \mathcal{O}_\xi)$ such that $(s) = 0$. By definition, for any collection $\{s_\alpha\}$ representing s we have $s_\alpha = \xi_{\alpha\beta} s_\beta$. Since s_β and s_α are nowhere vanishing we have $\xi_{\alpha\beta} = s_\alpha / s_\beta$, whence $(\delta\{s_\alpha\})_{\alpha\beta} = s_\beta s_\alpha^{-1} = \xi_{\beta\alpha}$ so that $\xi = \delta s$. Hence $\xi = 0$ in $H^1(X, \mathcal{O}_\xi)$. □

It follows that for a nontrivial bundle ξ any holomorphic section has at least one zero. We often refer to a holomorphically trivial bundle simply as a trivial bundle, but it is important to keep in mind that there are several notions of triviality, see also our discussion of Chern classes in Section 2.2.3. In order to investigate global sections further, and in particular to study existence of nontrivial global sections, it is useful to consider line bundles of the type $\xi = \delta_* D$ for some $D \in \Gamma(X, \mathcal{D})$. We denote its sheaf of sections by $\mathcal{O}_D := \mathcal{O}_{\delta_* D}$. This has a useful reformulation as follows. Consider the subsheaf $\mathcal{O}_{\mathcal{M},D}$ which we define as follows. For any point $p \in X$ we let the stalk be given by

$$(\mathcal{O}_{\mathcal{M},D})_p = \{\varphi = [U, f] \in \mathcal{M}_p \mid \text{ either } f \equiv 0 \text{ or } (f) \geqslant D|_U\}$$

and let $\mathcal{O}_{\mathcal{M},D} = \bigcup_{p \in X} (\mathcal{O}_{\mathcal{M},D})_p$. This defines obviously a subsheaf of $\mathcal{M}$ and we have the

57. Proposition [Gu, Lemma 7]**.** *The sheaves $\mathcal{O}_D$ and $\mathcal{O}_{\mathcal{M},D}$ are isomorphic. In particular, we can consider the sheaves of sections $\mathcal{O}_\xi$ as subsheaves of $\mathcal{M}$ for any line bundle ξ.*

Proof. We consider again the exact sequence $0 \to \mathcal{O}^* \to \mathcal{M}^* \to \mathcal{D} \to 0$ with induced boundary operator δ_*. Any divisor $D \in \Gamma(X, \mathcal{D})$ is locally induced by a meromophic function, that is, there exists an open covering $\mathcal{U} = \{U_\alpha\}$ together with meromorphic functions $d_\alpha \in \mathcal{M}^*_{U_\alpha}$ such that $(d_\alpha) = D|_{U_\alpha}$ and $\xi_{\alpha\beta} = d_\beta / d_\alpha$ is the cocycle defined by $\delta_* D$. To define the isomorphism $\mathcal{O}_{\mathcal{M},D} \to \mathcal{O}_D$ take a germ $f \in (\mathcal{O}_{\mathcal{M},D})_p$ with which we associate the meromophic germ $f_\alpha := f/d_\alpha$ if $p \in U_\alpha$. Since $(f_\alpha) = (f) - (d_\alpha) \geqslant 0$ this germ is necessarily holomorphic at p. Furthermore, if $p \in U_\alpha \cap U_\beta$ then $f_\alpha = f/d_\alpha = f \cdot \xi_{\alpha\beta}/d_\beta = \xi_{\alpha\beta} f_\beta$. In particular, f_α defines a function germ in $(\mathcal{O}_D)_p$, and the resulting map is easily seen to be an isomorphism. □

58. Corollary. *The nontrivial global sections of $\mathcal{O}_D = \delta_* D$ correspond precisely to meromorphic functions $f \in \mathcal{M}^*$ with $(f) \geqslant D$.*

It is natural to ask when a line bundle ξ is of the form $\delta_* D$. Obviously, the obstruction against this is the cohomology module $H^1(X, \mathcal{M}^*)$. Its vanishing has an interesting interpretation in terms of $\mathcal{M}_\xi$, the sheaf of **meromorphic cross sections** which is constructed in the same way as $\mathcal{O}_\xi$. For a cross section $s \in \Gamma(X, \mathcal{M}_\xi)$ we can define the **order** of s at p by setting $o_p(s) = o_p(s_\alpha)$ if $p \in U_\alpha$. Since s_α is well-defined up to a nowhere vanishing holomorphic function which has order 0, this is well-defined. For any not identically vanishing section s we can therefore define the **divisor of the cross section** s by

$$(s) = \sum_{p \in X} o_p(s) p.$$

In particular, $\Gamma(X, \mathcal{O}_\xi) = \{s \in \Gamma(X, \mathcal{M}_\xi) \mid (s) \geqslant 0\}$.

59. Lemma. $H^1(X,\mathcal{M}^*)=0 \Leftrightarrow$ *for all* $\xi\in H^1(X,\mathcal{O}^*)$, $H^0(X,\mathcal{M}_\xi)\neq\{0\}$, *that is,* ξ *admits a global meromorphic section.*

Proof. As for holomorphic sections on eshows that $\xi=[1]$ in $H^1(X,\mathcal{M}^*)\Leftrightarrow$ there exists $s\in\Gamma(X,\mathcal{M}^*_\xi)$, i.e. s is not identically zero $\Leftrightarrow\Gamma(X,\mathcal{M}_\xi)\neq 0$. By the exact sequence $0\to\mathcal{O}^*\xrightarrow{\iota}\mathcal{M}^*\to\mathcal{D}\to 0$ we can represent write every element $\eta\in H^1(X,\mathcal{M}^*)$ as $\eta=\iota_*\xi$ for $\xi\in H^1(X,\mathcal{O}^*)$. It follows that $H^1(X,\mathcal{M}^*)=0$ if and only if for every line bundle $\xi\in H^1(X,\mathcal{O}^*)$, $\Gamma(X,\mathcal{M}_\xi)\neq\{0\}$. □

We will see later that indeed $H^1(X,\mathcal{M}^*)=0$ for any compact Riemann surface, cf. Theorem 2.74. This basic analytic existence result is at the heart of the theory of Riemann surfaces. For the moment we continue to study the cohomology of $\mathcal{O}_\xi$. Towards that end we want to generalise the Dolbeault sequence $0\to\mathcal{O}\to\mathcal{A}^{0,0}\xrightarrow{\bar\partial}\mathcal{A}^{1,0}\to 0$ to an acyclic resolution of $\mathcal{O}_\xi$. Namely, in the same vein as $\mathcal{O}_\xi$ and $\mathcal{M}_\xi$ we can define **smooth cross sections** C^∞_ξ of ξ or **ξ-valued (p,q)-forms** $\mathcal{A}^{p,q}_\xi$. For instance, taking an atlas $\xi_{\alpha\beta}$ of ξ we define

$$\mathcal{A}^{p,q}_\xi=\Gamma(U_\alpha,\mathcal{A}^{p,q}_X)=\mathcal{A}^{p,q}_X(U_\alpha).$$

For $U_\beta\subset U_\alpha$ we take again the restriction functions provided by $\xi_{\beta\alpha}$, i.e. $\rho_{\alpha\beta}:\mathcal{A}^{p,q}_\xi(U_\alpha)\to\mathcal{A}^{p,q}_\xi(U_\beta)$ is given by $\rho_{\alpha\beta}(\varphi_\alpha)(p)=\xi_{\beta\alpha}(p)\cdot\varphi_{\alpha,p}$. For instance, if $\varphi_\alpha=f_\alpha dz_\alpha\in\mathcal{A}^{1,0}(U_\alpha)$ for some local coordinate z_α on U_α, then $\rho_{\beta\alpha}(\varphi_\alpha)(p)=\xi_{\alpha\beta}(p)\cdot f_\alpha(p)d_pz_\alpha$. The resulting presheaves are complete and fine, and they reduce to the usual sheaves if ξ is trivial. Furthermore, global sections are given locally as above by forms φ_α which satisfy $\varphi_\beta=\xi_{\beta\alpha}\varphi_\alpha$. Note that the exterior differential $d:\mathcal{A}^p_X\to\mathcal{A}^{p+1}_X$ does not induce a map $\mathcal{A}^p_\xi\to\mathcal{A}^{p+1}_X$ for the holomorphic functions $\xi_{\alpha\beta}:U_{\alpha\beta}\to\mathbb{C}^*$ do not commute with d, for $d\xi_{\alpha\beta}=\partial\xi_{\alpha\beta}$. However, they commute with $\bar\partial$ so that we can consider the **Dolbeault-Serre sequence**

$$0\longrightarrow\mathcal{O}_\xi\longrightarrow\mathcal{A}^{0,0}_\xi\xrightarrow{\bar\partial}\mathcal{A}^{0,1}_\xi\longrightarrow 0.$$

Since in a coordinate neighbourhood this reduces to the ordinary Dolbeault sequence (9) this sequence is exact and defines an acyclic resolution of $\mathcal{O}_\xi$. We immediately deduce from this the

60. Theorem (Dolbeault-Serre) [Gu, Theorem 8]**.** *Let X be a (not necessarily compact) Riemann surface, and $\xi\in H^1(X,\mathcal{O}^*)$ be a holomorphic line bundle* $\Rightarrow$

$$H^1(X,\mathcal{O}_\xi)\cong\Gamma(X,\mathcal{A}^{0,1}_\xi)/\bar\partial\Gamma(X,\mathcal{A}^{0,0}_\xi)$$
$$H^q(X,\mathcal{O}_\xi)=0,\quad q\geqslant 2.$$

It remains to investigate $H^q(X,\mathcal{O}_\xi)$ in more detail for $q=0$, 1. As a first step, we need that $H^0(X,\mathcal{O}_\xi)$ and $H^1(X,\mathcal{O}_\xi)$ are finite dimensional.

61. Theorem [Gu, Section 4.c]**.** *For all* $\xi\in H^1(X,\mathcal{O}^*)$, $\dim_{\mathbb{C}}H^q(X,\mathcal{O}_\xi)<\infty$ *if* $q=0$, 1.

Proof. (Sketch for $q=0$) The main idea is to show that $H^(X,\mathcal{O}_\xi)$ can be identified with a locally compact Hilbert space. Then $H^q(X,\mathcal{O}_\xi)$ must be finite dimensional by general arguments from functional analysis.

Step 1. *Define a Hilbert space structure on $\mathcal{O}$.* One first considers the square integrable holomorphic functions

$$\Gamma_0(U, \mathcal{O}) = \{f \in \mathcal{O}(U) \mid \int_U |f(z)|^2 dz \wedge d\bar{z} < \infty\}$$

and shows that this defines a Hilbert space. Furthermore, one can prove that the restriction operators $\rho_{UV} : \Gamma_0(U, \mathcal{O}) \to \Gamma_0(V, \mathcal{O})$ are bounded linear operators between Hilbert spaces which are *compact*, if $\bar{V}$ is compact and contained in U.

Let $\mathcal{U} = \{U_\alpha\}$ be an atlas of ξ. Out of Γ_0 we can construct the differential complex $\delta : C_0^q(\mathcal{U}, \mathcal{O}_\xi) \to C_0^{q+1}(\mathcal{U}, \mathcal{O}_\xi)$ between Hilbert spaces. We consider the associated cohomology groups which will be denoted by $H_0^q(\mathcal{U}, \mathcal{O}_\xi)$.

Step 2. *We have $H_0^q(\mathcal{U}, \mathcal{O}_\xi) \cong H^q(\mathcal{U}, \mathcal{O}_\xi) \cong H^q(X, \mathcal{O}_\xi)$.* This is essentially an application of Leray's theorem as $H^q(|\sigma|, \mathcal{O}_\xi) \cong H^q(|\sigma|, \mathcal{O})(= 0$ for $q \geqslant 1)$.

Step 3. *Finally, we show that $H_0^q(\mathcal{U}, \mathcal{O}_\xi)$ are Hilbert spaces and that refinement maps induce bounded compact operators between cohomology and that $H_0^q(\mathcal{U}, \mathcal{O}_\xi)$ is a locally compact Hilbert space, hence finite-dimensional.* Again, this follows essentially from functional analytic considerations.

□

In particular, the difference $\dim_{\mathbb{C}} H^0(X, \mathcal{O}_\xi) - \dim_{\mathbb{C}} H^1(X, \mathcal{O}_\xi)$ is well-defined. That this is a computable topological quantity is the content of the famous

62. Theorem (Riemann-Roch) [Gu, Theorem 13]**.** *Let X be a compact Riemann surface of genus g, and let $\xi \in H^1(X, \mathcal{O}^*)$ be a complex line bundle $\Rightarrow$*

$$\dim_{\mathbb{C}} H^0(X, \mathcal{O}_\xi) - \dim_{\mathbb{C}} H^1(X, \mathcal{O}_\xi) = 1 - g + c(\xi),$$

where $c(\xi) = \sum_{p \in X} o_p(s)$ is the so-called Chern class *of ξ given as the sum of all orders of a nontrivial meromorphic section s of ξ (whose existence has yet to be justified). In particular, if $c(\xi) > g - 1$, then $\mathcal{O}_\xi$ admits non-trivial holomorphic sections.*

Serre duality. While $H^0(X, \mathcal{O}_\xi)$ has an easy interpretation as the space of global sections, the module $H^1(X, \mathcal{O}_\xi)$ lacks such a simple interpretation. This makes the computation of its dimension quite difficult in general. However, in the special case of Riemann surfaces, this can be expressed in terms of the dimension of the space of global sections of some further sheaf:

63. Theorem (Serre duality) [Gu, Theorem 9]**.** *Let X be a compact Riemann surface, and let $\xi \in H^1(X, \mathcal{O}^*)$ be a holomorphic line bundle $\Rightarrow$ There is a natural isomorphism*

$$H^1(M, \mathcal{O}_\xi) \cong H^0(X, \Omega^1_{\xi^{-1}})^*.$$

In particular, $\dim_{\mathbb{C}} H^1(M, \mathcal{O}_\xi) = \dim_{\mathbb{C}} H^0(X, \Omega^1_{\xi^{-1}})^*$.

Recall that Ω^1 is the sheaf of holomorphic 1-forms, sometimes also called the *sheaf of abelian differentials*. The proof is functional analytic in nature, the interested read might consult [Gu, Chapter 6]. We merely outline the duality between these two spaces. First note that we have a natural bilinear pairing

$$\Gamma(X, \mathcal{A}^{0,1}_\xi) \times \Gamma(X, \mathcal{A}^{1,0}_{\xi^{-1}}) \to \Gamma(X, \mathcal{A}^{1,1}_X), \quad (\varphi, \psi) \mapsto \varphi \wedge \psi$$

(write out what this means in local coordinates – since we multiply φ_α and ψ_α by $\xi_{\beta\alpha}$ and $\xi_{\beta\alpha}^{-1}$ when passing to φ_β and ψ_β these contributions cancel and we can glue the locally defined forms $(1,1)$-forms to a global one). This $(1,1)$-form gives a complex number by integration (X is compact so we do not worry about convergence issues):

$$\Gamma(X, \mathcal{A}^{0,1}_\xi) \times \Gamma(X, \mathcal{A}^{1,0}_{\xi^{-1}}) \to \Gamma(X, \mathcal{A}^{1,1}_X), \quad (\varphi, \psi) \mapsto \langle \varphi, \psi \rangle := \int_X \varphi \wedge \psi.$$

Now if $\partial f \in \partial\Gamma(X, \mathcal{A}^{0,0}_\xi) \subset \Gamma(X, \mathcal{A}^{0,1}_\xi)$ and $\psi \in \Gamma(X, \Omega^1_{\xi^{-1}}) \subset \Gamma(X, \mathcal{A}^{1,0}_{\xi^{-1}})$ so that $\bar\partial\varphi = 0$ and $\partial(f\psi) \in \mathcal{A}^{2,0}(X) = 0$, we have

$$\langle \bar\partial f, \psi \rangle = \int_X \bar\partial f \wedge \psi = \int_X d(f\psi) = 0$$

by Stokes theorem (see Paragraph 2.15). The pairing $\langle \cdot, \cdot \rangle$ thus descends to

$$H^1(X, \mathcal{O}_\xi) \times H^0(X, \Omega^1_{\xi^{-1}}) \to \mathbb{C}, \quad ([\varphi], \psi) \mapsto \langle \varphi, \psi \rangle.$$

Serre's assertion is that this map is non degenerate. Note that while the isomorphism in Theorem 2.63 is natural, the isomorphism $H^0(X, \Omega^1_{\xi^{-1}})^* \cong H^0(X, \Omega^1_{\xi^{-1}})$ (which exists by finite dimensionality) is not.

To conclude this section we reformulate Serre's duality theorem in terms of the *canonical line bundle* K_X which avoids the use of differential forms by interpreting these as sections of K_X. As the name suggests the fact that K_X exists naturally on any Riemann surface and is (at least for X compact) nontrivial makes this a particularly interesting line bundle. To define it let $\mathcal{U}$ be a maximal atlas of X consisting of local charts $z_\alpha : U_\alpha \to \mathbb{C}$. We denote by $f_{\alpha\beta} : z_\beta(U_{\alpha\beta}) \to z_\alpha(U_{\alpha\beta})$ the resulting transition functions, i.e. we have

$$z_\alpha(p) = f_{\alpha\beta}(p)(z_\beta(p))$$

for all $p \in U_{\alpha\beta}$. We then define a cocycle $\kappa_{\alpha\beta} \in Z^1(\mathcal{U}, \mathcal{O}^*)$ by

$$\kappa_{\alpha\beta}(p) = \frac{1}{f'_{\alpha\beta}(z_\beta(p))}.$$

Indeed, if $p \in U_{\alpha\beta\gamma}$, then $z_\alpha(p) = f_{\alpha\gamma}(z_\gamma(p)) = f_{\alpha\beta}(f_{\beta\gamma}(z_\gamma(p)))$ so that by the chain rule for holomorphic functions,

$$\begin{aligned}
\kappa_{\alpha\gamma}(p) &= \left[f'_{\alpha\gamma}(z_\gamma(p))\right]^{-1} \\
&= \left[f'_{\alpha\beta}(f_{\beta\gamma}(z_\gamma(p))) \cdot f'_{\beta\gamma}(z_\gamma(p))\right]^{-1} \\
&= \left[f'_{\alpha\beta}(z_\beta(p)) \cdot f'_{\beta\gamma}(z_\gamma(p))\right]^{-1} \\
&= \kappa_{\alpha\beta}(p) \cdot \kappa_{\beta\gamma}(p).
\end{aligned}$$

64. Definition (canonical line bundle). We call the holomorphic line bundle $\kappa \in H^1(X, \mathcal{O}^*)$ the **canonical line bundle**.

Next we consider the sheaf Ω^1 of holomorphic 1-forms. In terms of the maximal atlas $\mathcal{U}$, a section $\varphi_\alpha \in \Omega^1(U_\alpha)$ is given by $\varphi_\alpha = g_\alpha dz_\alpha$ with $g_\alpha \in \mathcal{O}(U_\alpha)$ and $g_\alpha(p)d_pz_\alpha = g_\beta(p)d_pz_\beta$ if $p \in U_{\alpha\beta}$. Since $d_pz_\alpha = f'_{\alpha\beta}(z_\beta(p))d_pz_\beta = \kappa^{-1}_{\alpha\beta}(p)d_pz_\beta$, this means that

$$g_\alpha = \kappa_{\alpha\beta}g_\beta$$

so that $\{g_\alpha\}$ defines a section of κ. Hence $\Omega^1 = \mathcal{O}_\kappa$ and more generally, $\Omega^1_\xi = \mathcal{O}_{\kappa\xi}$ (where the product $\kappa\xi$ in $H^1(X, \mathcal{O}^*)$ is represented by the product cocycle $\kappa_{\alpha\beta}\cdot\xi_{\alpha\beta}$).

Similarly, $\mathcal{A}^{1,0}_\xi = \mathcal{A}^{0,0}_{\kappa\xi} = C^\infty_{\kappa\xi}$, the smooth sections of the holomorphic line bundle $\kappa\xi$. With this notation, we can restate Serre duality as

$$H^1(X, \mathcal{O}_\xi) \cong H^0(X, \mathcal{O}_{\kappa\xi^{-1}})^*.$$

2.3. **Statement and proof of Riemann-Roch.** We are now prepared to prove Theorem 2.62. We start by defining Chern classes, before we turn to the proof of Riemann-Roch.

Chern classes. We have seen in Lemma 2.56 that a line bundle is holomorphically trivial $\Leftrightarrow$ there exists a global section $s \in \mathcal{O}^*_X(X)$. More generally, we could ask for a smooth nowhere vanishing section, i.e. a family $\{f_\alpha\}$ with $f_\alpha \in C^{\infty *}(U_\alpha)$ such that $f_\alpha = \xi_{\alpha\beta} f_\beta$. If it exists we say that ξ is **smoothly trivial**. Geometrically, this means that the corresponding line bundle $L \to X$ is isomorphic to $X \times \mathbb{C}$ via a smooth (instead of a holomorphic) map. Still, such smooth trivialisations may not exist and we define obstructions against its existence via Čech cohomology. This will eventually lead to the definition of the Chern class of a holomorphic line bundle.

To start with we consider the exponential sequence $0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0$. This leads to the exact sequence $H^1(M, \mathbb{Z}) \to H^1(M, \mathcal{O}) \to H^1(M, \mathcal{O}^*) \to H^2(M, \mathbb{Z}) \to 0$ for $H^2(X, \mathcal{O}) = 0$ by Corollary 2.44. Consequently, the sequence

$$0 \longrightarrow H^1(M, \mathcal{O})/H^1(M, \mathbb{Z}) \longrightarrow H^1(M, \mathcal{O}^*) \xrightarrow{\delta_*} H^2(M, \mathbb{Z}) \longrightarrow 0$$

is exact.

65. Definition (Chern class). We call $c := \delta_*$ the **characteristic map** and $c(\xi) := \delta_*\xi$ the **Chern class of the line bundle** ξ.

To understand its geometric meaning, consider the smooth exponential sequence $0 \to \mathbb{Z} \to C^\infty \to C^{\infty *} \to 0$. The inclusions $\iota : \mathcal{O} \to C^\infty$ etc. give rise to the commutative diagramm

$$\begin{tikzcd}
H^1(X, \mathcal{O}) \arrow[r] \arrow[d, "\iota_*"] & H^1(X, \mathcal{O}^*) \arrow[r, "c"] \arrow[d, "\iota_*"] & H^2(X, \mathbb{Z}) \arrow[r] \arrow[d, "\cong"] & 0 \\
H^1(X, C^\infty) \arrow[r] & H^1(X, C^{\infty *}) \arrow[r, "c"] & H^2(X, \mathbb{Z}) \arrow[r] & 0
\end{tikzcd}$$

Now C^∞ is a fine sheaf so that $H^1(X, C^\infty) \cong H^2(X, \mathbb{Z})$. It follows that the *complex line bundle* associated with $\xi \in H^1(X, C^{\infty *})$, that is, we glue the local models $U_\alpha \times \mathbb{C}$ via the smooth transition functions $\xi_{\alpha\beta} : U_{\alpha\beta} \to \mathbb{C}^*$, is actually classified by $c(\xi)$. In particular, $c(\xi) = 0 \Leftrightarrow$ there exists a global section $\{f_\alpha\}$ of $C^{\infty *}_\xi(X)$ with $f_\alpha = \xi_{\alpha\beta} f_\beta$. For a *holomorphic line bundle* $\xi \in H^1(X, \mathcal{O}^*)$, $c(\xi) = 0$ means $\iota_*(\xi) = 0$ by the commutativity of the diagramm, that is, $\iota_*(\xi)_{\alpha\beta} = f_\beta/f_\alpha$ for smooth $f_\alpha \in C^{\infty *}(U_\alpha)$.

66. Remark.

(i) Similarly, we can topologically trivialise $\xi \Leftrightarrow$ there exists a global nowhere vanishing continuous section of ξ. Since we can approximate such a continuous section arbitrarily closely by a smooth section, a bundle ξ is topologically trivial $\Leftrightarrow \xi$ is smoothly trivial. In this sense, $c(\xi)$ is a topological invariant of ξ which classifies ξ completely as a complex vector bundle. Note, however, that even the smoothly trivial bundle $X \times \mathbb{C}$ might carry nontrivial holomorphic structures.

(ii) The previous discussion holds for any, in particular, noncompact Riemann surface. Since in this case $H^2(X,\mathbb{Z}) = 0$ we immediately deduce that any holomorphic line bundle is smoothly trivial. In fact, though this is beyond the scope of our course, one can show that it is even holomorphically trivial, see [Fo, 30.3].

Henceforth we restrict our attention again to compact Riemann surfaces. Then $H^2(X,\mathbb{Z}) \cong \mathbb{Z}$ so that $c(\xi)$ can be considered as an integer. We will give two different descriptions of this integer. The first is based on the inclusion $0 \to H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{C})$ induced by the inclusion $\mathbb{Z} \subset \mathbb{C}$. In general, the latter inclusion only gives a map $H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{C})$ which is not necessarily injective. From the *universal coefficient theorem*, however, it follows that $H^2(X,\mathbb{C}) \cong H^2(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}$. On the other hand, de Rham's theorem asserts that $H^2(X,\mathbb{C}) \cong \mathcal{A}^2(X)/d\mathcal{A}^1(X)$ so that we can represent $c(\xi)$ by a differential form $\varphi(\xi) \in \mathcal{A}^2(X)$. Integration of this yields the isomorphism $H^2(X,\mathbb{C}) \cong \mathbb{C}$, since $\int_X \varphi$ is independent of the representative by Stokes' Theorem. This form $\varphi(\xi)$ can be made explicit. In the sequel, let $C^\infty_{\mathbb{R}}$ denote the sheaf of *real-valued* smooth functions.

67. Proposition [Gu, Lemma 14]**.** *Let $\xi \in H^1(X,\mathcal{O}^*)$ be represented by $\{\xi_{\alpha\beta}\} \in Z^1(\mathcal{U},\mathcal{O}^*)$, and let $\{h_\alpha\} \in C^0(\mathcal{U},C^{\infty *}_{\mathbb{R}}\}$ be such that $h_\alpha(p) = |\xi_{\alpha\beta}(p)|^2 h_\beta(p)$ for $p \in U_{\alpha\beta}$. Then*

$$\varphi_\alpha = \frac{1}{2\pi i}\partial\bar{\partial}\log h_\alpha$$

defines a smooth 2-form such that

$$c(\xi) = \int_X \varphi.$$

68. Remark. Without loss of generality we can choose $h_\alpha > 0$. The transformation law $h_\alpha(p) = \bar{\xi}_{\alpha\beta}(p)h_\beta(p)\xi_{\beta\alpha}(p)$ is that of a **hermitian metric**: In general, if $h = (h_{ij})$ defines a hermitian metric on $\mathbb{C}^n$ in terms of a given basis, then a base change implemented by the matrix $A = (A_{ij}) \in \mathrm{GL}(n,\mathbb{C})$ is given by $\bar{A}^\top \cdot h \cdot A$. Given a hermitian metric on ξ induces a hermitian metric on the corresponding line bundle L by $h_p(v,w) = vh_\alpha(p)\bar{w}$ for v, $w \in L_p$. Now a hermitian metric always exists. Indeed, $\{|\xi_{\alpha\beta}|^2\}$ defines a cocycle in $Z^1(\mathcal{U},C^{\infty *}_{\mathbb{R}})$. Since the real exponential $\exp : C^\infty_{\mathbb{R}} \to C^{\infty *}_{\mathbb{R}}$ defines a sheaf isomorphism and $C^\infty_{\mathbb{R}}$ is obviously fine, $0 = H^1(X,C^\infty_{\mathbb{R}}) = H^1(X,C^{\infty *}_{\mathbb{R}})$. (In particular, any real line bundle – obtained by glueing the local models $U_\alpha \times \mathbb{R}$ – must be trivial) It follows that the cocycle $\{|\xi_{\alpha\beta}|^2\}$ must be a coboundary, i.e. equal to $\delta\{h_\alpha\}_{\alpha\beta} = h_\beta/h_\alpha$. The 2-form φ can then be interpreted as the *curvature* of the hermitian metric. For the special case of the holomorphic line bundle $-\kappa$ this has an interpretation in terms of the Gaussian curvature of a Riemannian (sic!) surface, and Riemann-Roch then essentially becomes the famous Gauß-Bonnet theorem, cf. also 2.77 (iii)

The relationship between hermitian metrics on complex vector bundles and topological invariants, the so-called *characteristic classes*, is at the heart of *Chern-Weil theory.*

Proof. The result will follow from a detailed analysis of the characteristic map $\delta_* : H^1(X,\mathcal{O}^*) \to H^2(X,\mathbb{Z})$ and the de Rham isomorphism $H^2(X,\mathbb{C}) \cong \mathcal{A}^2(X)/d\mathcal{A}^1(X)$.

Consider the exact sequence $0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0$ and represent ξ by a cocycle $\xi_{\alpha\beta}$ such that $U_{\alpha\beta}$ is simply-connected if nonempty (take an atlas of ξ and refine further). In particular, we can take the logarithm and obtain a cochain $\sigma_{\alpha\beta} = (\log \xi_{\alpha\beta})/2\pi i$ so that $\exp(\sigma_{\alpha\beta}) = \xi_{\alpha\beta}$. By definition, $c(\xi) = \delta_*\xi = [c_{\alpha\beta\gamma}]$ where $c_{\alpha\beta\gamma} \in Z^2(\mathcal{U}, \mathbb{Z})$ is the coboundary $c_{\alpha\beta\gamma} = (\delta\sigma)_{\alpha\beta\gamma} = \sigma_{\beta\gamma} - \sigma_{\alpha\gamma} + \sigma_{\alpha\beta}$. In order to represent this class by a 2-form we take the standard acyclic resolution of $\mathbb{C}$ to compute $H^2(X, \mathbb{C})$, namely

$$0 \longrightarrow \mathbb{C} \longrightarrow C^\infty \xrightarrow{d} \mathcal{A}^1 \xrightarrow{d} \mathcal{A}^2 \longrightarrow 0.$$

This gives rise to the exact sequence $0 \to \mathbb{C} \to \mathcal{A}^0 \to \mathcal{A}^1_d \to 0$ where $\mathcal{A}^1_d$ denotes the closed 1-forms (this corresponds to the sequence $0 \to \mathcal{F} \to \mathcal{F}_0 \to \mathcal{K}_1 \to 0$ in Theorem 2.38). Regarding $c_{\alpha\beta\gamma}$ as a cocycle in $C^2(\mathcal{U}, \mathbb{C})$ it is the coboundary of the cocycle $\sigma_{\alpha\beta}$ in $C^1(\mathcal{U}, C^\infty)$. In particular, $d\sigma_{\alpha\beta} \in Z^1(\mathcal{U}, \mathcal{A}^1_d)$. On the other hand, the exact sequence $0 \to \mathcal{A}^1_d \to \mathcal{A}^1 \to \mathcal{A}^2 \to 0$ (corresponding to $0 \to \mathcal{K}_1 \to \mathcal{F}_1 \to \mathcal{K}_2 \to$ in Theorem 2.38) shows that $d\sigma_{\alpha\beta}$, regarded as an element in $Z^1(\mathcal{U}, \mathcal{A}^1)$ via the inclusion $0 \to \mathcal{A}^1_d \to \mathcal{A}^1$, must be the coboundary of a cochain in $C^0(\mathcal{U}, \mathcal{A}^1)$ for $H^1(X, \mathcal{A}^1) = 0$, $\mathcal{A}^1$ being fine. It follows that we need to find $\tau_\alpha \in C^0(\mathcal{U}, \mathcal{A}^1)$ such that $d\sigma_{\alpha\beta} = (\tau_\alpha - \tau_\beta)|_{U_{\alpha\beta}}$, or equivalently, we need a family of smooth 1-forms $\tau_\alpha \in \mathcal{A}^1(U_\alpha)$ satisfying

$$\tau_\alpha = \frac{1}{2\pi i} d \log \xi_{\alpha\beta} + \tau_\beta \quad \text{in} \quad U_\alpha \cap U_\beta.$$

In particular, $d\tau_\alpha = d\tau_\beta$ on $U_{\alpha\beta}$ so that $\varphi_\alpha = d\tau_\beta$ pieces together to a well-defined 2-form φ which represents the cohomology class $[c_{\alpha\beta\gamma}] \in H^2(X, \mathbb{C})$.

We can now conclude by constructing τ_α as follows. By assumption, $\log h_\alpha = \log h_\beta + \log \xi_{\alpha\beta} + \log \bar{\xi}_{\alpha\beta}$ over $U_{\alpha\beta}$. Since the functions $\xi_{\alpha\beta}$ are holomorphic, $d \log \xi_{\alpha\beta} = \partial \log \xi_{\alpha\beta}$ and $\partial \log \bar{\xi}_{\alpha\beta} = 0$. Taking $\tau_\alpha = i/2\pi \cdot \partial \log h_\alpha$ does the job, and $\varphi_\alpha = d\tau_\alpha = i/2\pi \bar{\partial}\partial \log h_\alpha$. ☐

Let $f \in H^0(X, \mathcal{M}^*_\xi)$ be a nontrivial meromorphic section of ξ. The integer

$$\deg(f) := \sum_{p \in X} o_p(f)$$

is called the **total order** or the **degree of** f. Since X is compact, $o_p(f) \neq 0$ only for a finite number of points p.

69. Theorem [Gu, Theorem 11]. *For any line bundle $\xi \in H^1(X, \mathcal{O}^*)$ over a compact Riemann surface X, and any nontrivial meromorphic section $f \in H^0(X, \mathcal{M}^*)$, we have*

$$c(\xi) = \sum_{p \in X} o_p(f).$$

In particular, any nontrivial meromorphic section has the same total order.

Proof. We are building a hermitian metric from f and apply the previous proposition. Namely, let $p_1, \ldots, p_n$ the finite number of points of X where $o_p(f) \neq 0$. We represent ξ by a cocycle $\xi_{\alpha\beta}$ where we choose the covering in such a way that for all $i = 1, \ldots, n$, there exists an open neighbourhood V_i of p_i contained in U_{α_i} for some α_i, but $V_i \cap U_\alpha = \varnothing$ if $\alpha \neq \alpha_i$. We have $f_\alpha = \xi_{\alpha\beta} f_\beta$ on $U_{\alpha\beta} \neq \varnothing$. We let $h_\alpha = |f_\alpha|^2 > 0$ on $U_\alpha \backslash \bigcup_i V_i$ and extend it smoothly and positively over $\bigcup_i V_i$. In particular, $h_\alpha = |\xi_{\alpha\beta}|^2 h_\beta$ so that h defines a hermitian metric on ξ. (Intiutively, f defines a trivialisation of ξ near p whenever $o_p(f = 0$; we then let $|f|^2$ be the norm of the trivialisation.)

Next we evaluate $c(\xi)$ using the previous proposition so that

$$c(\xi) = \frac{1}{2\pi i}\int_X \partial\bar{\partial}\log h_\alpha^{-1} = \frac{1}{2\pi i}\int_X \bar{\partial}\partial h_\alpha.$$

Outside $\bigcup_i V_i$ we have $h_\alpha = |f_\alpha|^2 = \bar{f}_\alpha \cdot f_\alpha$ so that

$$\bar{\partial}\partial \log h_\alpha = \bar{\partial}\partial(\log f_\alpha + \log \bar{f}_\alpha) = 0.$$

Indeed, $f_\alpha = f_\alpha(z)$ is holomorphic in z so that $\bar{\partial} f = 0$ and $\partial\bar{f} = \overline{\bar{\partial} f} = 0$. Hence

$$\begin{aligned} c(\xi) &= \frac{1}{2\pi}\sum_i \int_{V_i} \bar{\partial}\partial \log h_{\alpha_i} \\ &= \frac{1}{2\pi i}\sum_i \int_{V_i} d\partial \log h_{\alpha_i} \\ &= \frac{1}{2\pi i}\sum_i \int_{\partial V_i} \partial \log h_{\alpha_i} \end{aligned}$$

by Stokes' theorem. But $h_\alpha = |f_\alpha|^2$ on the boundary ∂V_i of V_i,, whence $\partial \log h_\alpha = (\log f_\alpha)' = f_\alpha'/f_\alpha$. Consequently,

$$c(\xi) = \frac{1}{2\pi i}\sum_i \int_{\partial V_i} \frac{f_\alpha'}{f_\alpha} = \sum_i o_{p_i}(f)$$

by the Rouché's formula A.31 (which is essentially the residue theorem). □

Since any global holomorphic section defines in particular a meromorphic one with nonnegative total order we immediately dedude the

70. Corollary. *If $c(\xi) < 0$ then there are no nontrivial global holomorphic sections of ξ, that is, $H^0(X, \mathcal{O}_\xi) = 0$.*

As a further corollary we note

71. Corollary. *If $\xi = \delta_* D$, $D = \sum a_i p_i \Rightarrow c(\xi) = -\deg D = -\sum a_i$.*

Proof. Indeed, if we represent D locally by $d_\alpha \in \mathcal{M}^*(U_\alpha)$ then $\xi_{\alpha\beta} = (\delta_* D)_{\alpha\beta} = d_\beta/d_\alpha$. Hence we obtain a meromorphic section $s = \{1/d_\alpha\}$ whose induced divisor is $(s) = -D$. □

Proof of Riemann-Roch. For the rest of this section X will be a compact Riemann surface. Let $\xi \in H^1(X, \mathcal{O}^*)$ be a holomorphic line bundle. We define

$$\begin{aligned} \tilde{\chi}(\xi) &= \dim H^0(X, \mathcal{O}_\xi) - \dim H^1(X, \mathcal{O}_\xi) - c(\xi) \\ &= \dim H^0(X, \mathcal{O}_\xi) - \dim H^0(X, \mathcal{O}_{\kappa\xi^{-1}}) - c(\xi), \end{aligned}$$

where we used Serre duality for the last line.

72. Remark. The integer

$$\chi(\xi) = \dim H^0(X, \mathcal{O}_\xi) - \dim H^1(X, \mathcal{O}_\xi) = \tilde{\chi}(\xi) + c(\xi)$$

is called the **(holomorphic) Euler characteristic** of ξ.

We know already that this quantity is well-defined. We will need to show that it is independent of $\tilde{\xi}$ and that it equals $\chi(\xi) = 1 - g$ where g is the genus of the surface. First we investigate the case of a divisor.

73. Lemma. *Let $D \in \mathcal{D}(X)$, and let $\eta = \delta_* D \in H^1(X, \mathcal{O}^*) \Rightarrow$*

$$\tilde{\chi}(\xi\eta) = \tilde{\chi}(\xi).$$

Proof. If $D = \sum a_i p_i$ then $\tilde{\chi}(\xi \cdot \delta_* D) = \tilde{\chi}(\Pi_i \xi \eta_i^{a_i})$ where $\eta_i = \delta_* p_i$. Hence it is sufficient to prove the assertion for a *point bundle* $\eta = \delta_* p$, $p \in X$. In analogy with the sheaves considered in Proposition 2.57 we introduce the subsheaf $\mathcal{O}_{\mathcal{M},\, D,\, \xi}$ of $\mathcal{M}_\xi$ defined by

$$(\mathcal{O}_{\mathcal{M},\, D,\, \xi})_q = \{[U, f] \in (M_\xi)_q \mid f \equiv 0 \text{ or } (f) \geqslant D|_U\}.$$

Since $D = p > 0$, $\mathcal{O}_{\mathcal{M},\, D,\, \xi}$ is actually a subsheaf of $\mathcal{O}_\xi$, and we can consider the quotient sheaf $\mathcal{Q} = \mathcal{O}_\xi / \mathcal{O}_{\mathcal{M},\, D,\, \xi}$. This is a skyscraper sheaf of the form

$$\mathcal{Q}_q = \begin{cases} 0 \text{ if } q \neq p \\ \mathbb{C} \text{ if } q = p \end{cases}$$

(the stalk $\mathcal{O}_{\mathcal{M},\, D,\, \xi}$ at p is just the maximal ideal $\mathfrak{m}$ of the local ring $(\mathcal{O}_\xi)_p$ given by noninvertible sections). In fact, $\mathcal{O}_{\mathcal{M},\, D,\, \xi} \cong \mathcal{O}_{\xi\eta}$ (this follows as in Proposition 2.57). From the resulting exact sequence $0 \to \mathcal{O}_{\xi\eta} \xrightarrow{\iota} \mathcal{O}_\xi \xrightarrow{\pi} \mathcal{S} \to 0$ we deduce the long exact sequence

$$0 \longrightarrow H^0(X, \mathcal{O}_{\xi\eta}) \longrightarrow H^0(X, \mathcal{O}_\xi) \xrightarrow{\pi_*} H^0(X, \mathcal{Q}) \xrightarrow{\delta_*}$$

$$\longrightarrow H^1(X, \mathcal{O}_{\xi\eta}) \xrightarrow{\iota_*} H^1(X, \mathcal{O}_\xi) \longrightarrow H^1(X, \mathcal{Q}) \longrightarrow 0 \longrightarrow \dots$$

Since $\mathcal{Q}$ is a skyscraper sheaf it is flabby and thus acyclic with $H^0(X, \mathcal{Q}) \cong \mathbb{C}$. From this and the exactness of the sequence it follows that

- $\dim \operatorname{im} \pi_* + \dim H^0(X, \mathcal{O}_{\xi\eta}) = \dim H^0(X, \mathcal{O}_\xi)$,
- $\dim H^1(X, \mathcal{O}_\xi) + \dim \ker \iota_* = \dim H^1(X, \mathcal{O}_{\eta\xi})$,
- $1 = \dim \operatorname{im} \pi_* + \dim \ker \iota_*$,

that is, we find for the alternating sum

$$\begin{aligned} &\dim H^0(X, \mathcal{O}_{\xi\eta}) - \dim H^1(X, \mathcal{O}_{\xi\eta}) + 1 \\ &= \dim H^0(X, \mathcal{O}_\xi) - \dim H^1(X, \mathcal{O}_\xi). \end{aligned}$$

Since $1 = -c(\eta) = c(\eta^{-1})$, adding $c(\xi)$ to both sides and using $c(\xi) + c(\eta^{-1}) = c(\xi\eta^{-1})$ gives the desired formula $\tilde{\chi}(\xi\eta^{-1}) = \tilde{\chi}(\xi)$. $\square$

As an important consequence we deduce from this and Lemma 2.59 the

74. Theorem [Gu, Theorem 12]**.** *On a compact Riemann surface X,*

$$H^1(X, \mathcal{M}^*) = 0,$$

i.e. every line bundle on M has a nontrivial meromorphic section and is thus of the form $\delta_* D$ for some divisor D uniquel determined up to (f) for $f \in \mathcal{M}_X^*(X)$.

Proof. Since a line bundle comes from a divisor if and only if it admits a meromorphic section, it is enough to show that there exists a line bundle $\eta = \delta_* D$ such that $\xi\eta$ admits a meromorphic section. We will actually be able to show that we can find a holomorphic line bundle η such that $\xi\eta$ has in fact a holomorphic section. Suppose to the contrary that $H^0(X, \mathcal{O}_{\xi\eta}) = 0$ for all $D \in \mathcal{D}(X)$. By the previous lemma,

$$\tilde{\chi}(\xi\eta) = \dim H^0(X, \mathcal{O}_{\xi\eta}) - \dim H^0(X, \mathcal{O}_{\kappa\xi^{-1}\eta^{-1}}) - c(\xi\eta)$$

is independent of D so that by our assumption,

$$\dim H^0(X, \mathcal{O}_{\kappa\xi^{-1}\eta^{-1}}) + c(\xi\eta) = c$$

for some constant independent of D. Choosing D suitably we can arrange for $c(\kappa\xi^{-1}\eta^{-1}) = c(\kappa) - c(\xi) - c(\eta) < 0$ so that $\kappa\xi^{-1}\eta^{-1}$ admits no nontrivial holomorphic sections by Corollary 2.70. But then $c(\eta) = c - c(\xi)$ is independent of D, a contradiction. $\square$

75. Corollary. *We have*

$$\mathbf{Cl}(X) \cong H^1(X, \mathcal{O}^*),$$

that is, the divisor class group is isomorphic to the group of holomorphic line bundles.

76. Corollary. *The number $\tilde{\chi}(\xi)$ is constant and does* not *depend on ξ.*

Proof. Since every line bundle ξ comes from a divisor, we have $\tilde{\chi}\xi) = \tilde{\chi}(1 \cdot \xi) = \chi(1)$. $\square$

In particular, we find

$$\tilde{\chi}(\xi) = \tilde{\chi}(1) = \dim H^0(X, \mathcal{O}) - \dim H^0(X, \mathcal{O}_\kappa) - c(1) = 1 - \dim H^0(X, \mathcal{O}_\kappa),$$

We call the constant

$$g := \dim H^1(X, \mathcal{O}_\kappa)$$

the **arithmetic genus** of X. To interpret g in terms of the topology underlying the Riemann surface we consider the exact sequence

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{O} \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 = 0$$

(a holomorphic 1-form $\omega \in \Omega^1$ is closed, hence locally of the form $\omega = f dz = dg$ for $g \in C^\infty$ – it is easy to see that g is holomorphic, hence the sequence is indeed exact). This gives rise to the long exact sequence

$$\begin{array}{l} 0 \longrightarrow H^0(X,\mathbb{C}) \longrightarrow H^0(X,\mathcal{O}) \longrightarrow H^0(X,\Omega^1) \longrightarrow \\ \longrightarrow H^1(X,\mathbb{C}) \longrightarrow H^1(X,\mathcal{O}) \longrightarrow H^1(X,\Omega^1) \longrightarrow \\ \longrightarrow H^2(X,\mathbb{C}) \longrightarrow 0 \longrightarrow \ldots \end{array}$$

Now $H^0(X, \mathbb{C}) \cong H^0(X, \mathcal{O}) \cong H^2(X, \mathbb{C}) \cong \mathbb{C}$, and as above it follows that

$$\dim H^0(X, \Omega^1) - \dim H^1(X, \mathbb{C}) + \dim H^1(X, \mathcal{O}) - \dim H^1(X, \Omega^1) + 1 = 0$$

By Serre Duality we have $\dim H^1(X, \mathcal{O}) = \dim H^0(X, \Omega^1) = \dim H^0(X, \mathcal{O}_\kappa) = g$ and $\dim H^1(X, \Omega^1) = \dim H^0(X, \mathcal{O}) = 1$. Consequently, $2g = \dim H^1(X, \mathbb{C})$ which means that g equals also the topological genus of X (while this is only define in the surface case, the arithmetic genus can be defined for arbitrary complex manifolds). It also follows that

$$\tilde{\chi}(\xi) = 1 - g$$

which completes the proof of Theorem 2.62.

77. Applications of Riemann-Roch. Finally, we discuss some applications of Riemann-Roch.

(i) *Existence of nontrivial holomorphic sections.* Given a line bundle ξ over X one is naturally interested in the holomorphic sections and their zeroes: Assume that there are nontrivial sections $s_0 \ldots, s_n \in H^0(X, \xi)$. Then $p \mapsto [s_0(p) : \ldots : s_n(p)] \in \mathbb{P}^n$ is well-defined whenever p is not a *base point*, i.e. $s_i(p) = 0$ for all p. Here, we think of the s_i as trivialised sections of the associated line bundle L_ξ. Of course, the $n+1$-tupel $(s_0, \ldots, s_n(p))$ depends on the trivialisation. But any other trivialisation multiplies this $n+1$-tuple by a nonzero complex number so that the line $[s_0(p) : \ldots : s_n(p)] \in \mathbb{P}^n$ is well-defined. It turns out that we that there are no base points and that the map $X \to \mathbb{P}^n$ is actually an embedding if there are sufficiently many sections. In fact, there exists an integer n so that considering ξ^n instead of n provides such a line bundle with sufficiently many sections (we also say that ξ is *ample*. To see how Riemann-Roch can be of help here, consider $\xi = \kappa$, the canonical line bundle.Then

$$\tilde{\chi}(\kappa) = \dim H^0(X, \mathcal{O}_\kappa) - \dim H^1(X, \mathcal{O}_\kappa) - c(\kappa) = g - 1 - c(\kappa)$$

so that

$$c(\kappa) = 2(g-1). \tag{10}$$

This implies not only that a holomorphic 1-form has at exactly $2(g-1)$ zeroes counted with mulitplicity, but also that

$$\dim H^0(X, \mathcal{O}_\kappa) \geqslant g - 1.$$

It follows that any Riemann surface of genus $\geqslant 2$ has nontrivial holomorphic 1-forms. We discuss further case of existence in the problem class.

(ii) *Dimension of the Teichmüller space.* Given a compact surface we can ask how many ineqivalent Riemann surface structures we can define on X. This question can be asked more generally for any (compact) manifold of arbitrary dimension. Specialised to the case of a Riemann surface X the answer is this. Fixing one particular Riemann structure the answer is this (cf. for instance [**?**, 6.1.5 and 6.1.6]): The space of possible Riemann surface structures is a smooth complex manifold $\mathcal{T}_X$ (the *Teichmüller space* of X) if and only if $H^2(X, \mathcal{O}_{\kappa^{-1}}) = 0$ (which is always the case, cf. Theorem 2.60), and in this case its dimension is $\dim H^1(X, \mathcal{O}_{\kappa^{-1}})$. To compute the latter we note that by Riemann-Roch

$$\dim H^1(X, \mathcal{O}_{\kappa^{-1}}) = 1 - g + c(\kappa^{-1}) - \dim H^0(X, \mathcal{O}_{\kappa^{-1}})$$

Now $c(\kappa^{-1}) = -c(\kappa) = 2(1-g) < 0$ for $g \leqslant 2$ which implies $H^0(X, \mathcal{O}_{\kappa^{-1}}) = 0$ by Corollary 2.70. Hence $\dim \mathfrak{T}_X = 3(g-3)$ if $g \geqslant 2$. That X carries at least one Riemann surface structure is of course nontrivial. This follows for instance from the fact that the fundamental group of any compact surface acts as biholmorphically via Deck transformations on the unviersal covering $\mathbb{C}$, cf. for instance also the discussion in [Gu, §9]. Summarising, we also yield a smooth Teichmller space which is of dimension 0 on the sphere, of dimension 1 on a torus, and of dimension $3(g-3)$ on a surface of genus $\geqslant 2$.

(iii) Expressing the Chern class $c(\kappa)$ as an integral over X as in Proposition 2.??, Riemann-Roch actually gives the famous *Gauss-Bonnet-Theorem* for Riemannian (sic!) surfaces, that is, surfaces equipped with a Riemannian metric. Indeed, underlying the real tangent space and ist Riemannian is the dual of the holomorphic line bundle κ together with a hermitian metric. The integrand $i\partial\bar\partial \log h_\alpha$ can then be interpreted as the Gaussian curvature of the Riemannian metric.

Appendix A. Holomorphic functions

We briefly recall the most important features of holomorphic functions. For a general reference see for instance [Ah] or [Re]. As a matter of notation, we let $D_\epsilon(z_0)$ be the open disc of radius ϵ around $z_0 \in \mathbb{C}$, i.e. $D_\epsilon(z_0) = \{z \in \mathbb{C} \mid |z - z_0| < \epsilon\}$.

1. Definition. Let $U \subset \mathbb{C} \cong \mathbb{R}^2$ be an open subset. A function $f : U \to \mathbb{C}$ is called **complex differentiable at** $a \in U$ or **holomorphic at** $a \in U$ if

$$f'(a) := \lim_{z\to a} \frac{f(z) - f(a)}{z - a}$$

exists. In this case, $f'(a)$ is called the **complex derivative** of f at a. If is holomorphic for every $a \in U$ we say that f is **holomorphic on** U and let $\mathcal{O}(U)$ be the set of holomorphic functions on U.

2. Example.

(i) $f : \mathbb{C} \to \mathbb{C}$, $f(z) \equiv c \in \mathbb{C}$ the constant functions are holomorphic on $\mathbb{C}$ with $f'(a) = 0$, whence $\mathbb{C} \subset \mathcal{O}(\mathbb{C})$ (identifying constant functions with their complex value).

(ii) $f : \mathbb{C} \to \mathbb{C}$, $f(z) = cz$, $c \in \mathbb{C}$ is holomorphic with $f'(a) = c$.

(iii) If $f \in \mathcal{O}(U)$ and $V \subset U$ is an open subset, then $f|_V \in \mathcal{O}(V)$. In particular, $\mathbb{C} \subset \mathcal{O}(U)$

Functions which are holomorphic on all of $\mathbb{C}$ are also called **entire**. As for the usual (real) differentiability we can prove in exactly the same way the following differentiation rules.

3. Proposition. *Let f, $g \in \mathcal{O}(U)$. Then*

(i) *$f + g \in \mathcal{O}(U)$, and $(f + g)'(a) = f'(a) + g'(a)$;*

(ii) *$f \cdot g$, and $(f \cdot g)'(a) = f'(a) \cdot g(a) + f(a) \cdot g'(a)$.*

In particular, $\mathcal{O}(U)$ is a $\mathbb{C}$-algebra.

(i) *If g is a function which is holomorphic in a, and f is holomorphic in $g(a)$, then $f \circ g$ is holomorphic in a, and we have*

$$(f \circ g)'(a) = f'(g(a)) \cdot g(a).$$

4. Example. Since $\mathcal{O}(U)$ is a $\mathbb{C}$-algebra and $z \in \mathcal{O}(U)$ by restriction, any polynomial $f(z) = \sum_{i=0}^n a_i z^i$ is in $\mathcal{O}(U)$, and $f'(a) = \sum_{i=0} i a_i z^{i-1}$. More generally, let $f : U \to \mathbb{C}$ be **analytic**, that is, for any $z_0 \in U$ exists an $\epsilon > 0$ such that $D_\epsilon(z_0) \subset U$ and

$$f(z) = \sum_{i=0}^\infty a_i (z - z_0)^i \qquad z \in D_\epsilon(z_0)$$

where the convergence is absolute and uniform in z. It follows that f is holomorphic and differentiating term by term yields

$$f'(z) = \sum_{i=0}^{\infty} i a_i (z - z_0)^{i-1} \qquad z \in D_\epsilon(z_0)$$

as in the real differentiable case. For instance, the complex exponential function

$$e^z := \sum_{i=0}^{\infty} \frac{z^i}{i!}$$

is entire.

In terms of real differentiability we can express the condition to be holomorphic as follows. Under the identification $\mathbb{C} \cong \mathbb{R}^2$, $z = x + iy$ corresponds to (x, y).

5. Theorem. *A function $f : U \to \mathbb{C}$ is holomorphic in $a \in U \Leftrightarrow f$ is real differentiable in a and the* **Cauchy-Riemann equations**

$$\partial_x u(a) = \partial_y v(a), \qquad \partial_y u(a) = -\partial_x v(a)$$

hold. In particular, any holomorphic function is continuous.

Proof. See for instance [Pö, 9.4]. □

In practice, we will use the previous theorem as follows. We let

$$\partial_z f(a) := \frac{1}{2}(\partial_x f - i\partial_y f)(a), \quad \partial_{\bar{z}} f(a) := \frac{1}{2}(\partial_x f + i\partial_y f)(a).$$

We also write ∂ for ∂_z and $\bar{\partial}$ for $\partial_{\bar{z}}$. Then we obtain the following

6. Corollary. *A function $f \in C^1(U)$ is holomorphic at $a \Leftrightarrow$*

$$\bar{\partial} f(a) = 0.$$

In this case, $f'(a) = \partial f(a)$.

7. Remark.

(i) Of course, the regularity assumption of Corollary A.6 is not optimal, but it will simpliyfy the subsequent discussion since where we use Stokes' Theorem to derive Cauchy's integral formula. At any rate, this will be sufficient for our purposes.

(ii) To see where the Cauchy-Riemann equations comes from, consider a smooth function $f(x, y) = u(x, y) + iv(x, y) : U \to \mathbb{C} \cong \mathbb{R}^2$. Its real derivative at a with respect to the real standard basis $e_1 = 1$, $e_2 = i$ of $\mathbb{R}^2$ is

$$D_a f = \begin{pmatrix} \partial_x u & \partial_y u \\ \partial_x v & \partial_y v \end{pmatrix}.$$

This is a *real* linear map, that is $D_a f(\lambda v) = \lambda D_a f(v)$ for all $v \in \mathbb{R}^2$, $\lambda \in \mathbb{R}$. When is it complex linear? Multiplication with i sends $e_1 = 1$ to $e_2 = i$ and $e_2 = i$ to $-1 = e_1$, so $D_a f$ is complex linear if and only it commutes with the corresponding matrix, i.e.

$$D_a f \circ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \circ D_a f.$$

It is easy to see that this holds $\Leftrightarrow$ the Cauchy-Riemann equations hold. Summarising, a smooth function $f : U \to \mathbb{C} \cong \mathbb{R}^2$ is holomorphic $\Leftrightarrow$ its differential is complex linear.

(iii) In general, a complex linear map $A : \mathbb{R}^2 \to \mathbb{R}^2$ is given by a matrix of the form

$$A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

which after the identification $\mathbb{R}^2 \cong \mathbb{C}$ corresponds to the complex linear map given by multiplication with $a+ib$. Hence, seeing $D_a f$ as a complex number if $D_a f$ is complex linear, i.e. f is holomorphic, we have $D_a f = \partial_x u - i\partial_y u = \partial f$.

The upshot of this ∂-$\bar\partial$ formalism is that we can treat a holomorpic function as a "differentiable" function in z whose derivative at a is given by $\partial f(a)$ and for which the usual differentiation rules hold.

8. Example.

(i) $f(z) = z^k$ is holomorphic for $\bar\partial f(a) = 0$ and $\partial f(a) = ka^{k-1}$.

(ii) The **exponential map** $\exp : \mathbb{C} \to \mathbb{C}^*$ defined by $\exp(z) = \sum_{k=0}^{\infty} z^k/k!$ is holomorphic (more generally, any power series which converges absolutely and uniformly on some open disc is holomorphic there). Note thta exp defines a group morphism if $\mathbb{C}$ and $\mathbb{C}^*$ are endowed with their usual group structures of addition and multiplication of complex numbers. In particular, $\ker \exp = 2\pi i\mathbb{Z}$.

(iii) If $f \in \mathcal{O}(U)$, then a **logarithm of** f is a function $g \in \mathcal{O}(U)$ such that $\exp(g) = f$. Of course, $f(z) \neq 0$ for all $z \in U$ is a necessary condition for a holomorphic logarithm to exist, but due to monodromy issues (cf. also Section 11.3) it is in general not sufficient. However, the logarithm exists if e.g. U is simply connected. In this case, one can also consider the **k-th holomorphic root** of f, namely $h \in \mathcal{O}(U)$ with $h^k = f$. Take, for instance, $h = \exp(g/k)$, where g is a holomorphic logarithm (see also [Re, Section 9.3] for instance).

(iv) $f(z) = |z|^2 = z \cdot \bar z$ is not holomorphic, for it depends on $\bar z$: $\bar\partial f(z) = z$.

(v) Let $f : U \to \mathbb{C}$, $g : V \to U \subset \mathbb{C}$ two smooth functions so that $f \circ g : V \to \mathbb{C}$ is also smooth. Then for all $a \in V$,

$$\partial(f \circ g)(a) = \partial f(g(a)) \cdot \partial g(a) + \bar\partial f(g(a)) \cdot \partial \bar g(a),$$
$$\bar\partial(f \circ g)(a) = \partial f(g(a)) \cdot \bar\partial g(a) + \bar\partial f(g(a)) \cdot \bar\partial \bar g(a)$$

(see for instance [Re, p.68]). In particular, if f and g are holomorphic, so is $f \circ g$ and we obtain the *holomorphic chain rule*

$$\partial(f \circ g)(a) = (f \circ g)'(a) = f'(g(a)) \cdot g'(a).$$

9. Remark. In analogy with real analysis we can also consider holomorphic functions in several variables. A function $f = (f_1, \ldots, f_m) : U \subset \mathbb{C}^n \to \mathbb{C}^m$ is holomorphic $\Leftrightarrow$ $f_i : U \to \mathbb{C}$ is contininuous and $f(z_1, \ldots, z_n)$ holomorphic in any single variable z_i in the sense above. Then many familiar theorems of real analysis still hold, in particular the *implicit function theorem* which allows to set up a theory of complex manifolds along the lines of differentiable manifolds. For instance this theorem for holomorphic functions $f : U \subset \mathbb{C}^2 \to \mathbb{C}$ reads as follows. If $f \equiv f(w, z)$ is holomorphic and if $\partial_z f(a) \neq 0$, then there exists a holomorphic function $g \equiv g(w)$ defined in some neighbourhood of a such that $f(w, g(w)) = f(a)$. Put differently, we can eliminate holomorphically the variable z from the equation $f(w, z) = f(a)$. See also [GuRo] for more details on holomorphic functions in severable variables.

Before we come to the next theorem we define

$$dz = dx + idy, \qquad d\bar{z} = dx - idy.$$

We can think of $(dz, d\bar{z})$ as the basis dual to $(\partial, \bar{\partial})$. In particular,

$$df = \partial f dz + \bar{\partial} f d\bar{z}.$$

The **volume element** $dx \wedge dy$ used for integration is then $idz \wedge \bar{d}z/2$. Recall also that for the line integral of a continuous f defined on the nighbourhood of a piecewise C^1 curve $\gamma : I \to \mathbb{C}$ is defined by

$$\int_\gamma f(z)dz := \int_I f(\gamma(t)) \cdot \gamma'(t)dt.$$

This is indepenent of the particular parametrisation of γ, and if we write $f = (u, v) = u + iv$ and $\gamma = (x, y)$, then $\int_\gamma f dz = \int_\gamma (udx - vdy) + i \int_\gamma (vdx + udy)$. We can now state the central theorem for holomorphic functions.

10. Theorem (Cauchy's integral formula). *Let $w \in D := D_\epsilon(z_0) \subset \mathbb{C}$ and $f \in C^1(\bar{D})$ (that is, f is continuously differentiable on D and continuous on $\bar{D}$) $\Rightarrow$*

$$f(w) = \frac{1}{2\pi i}\Big(\int_{\partial D} \frac{f(z)}{z - w} dz + \int_\Delta \frac{\bar{\partial} f(z)}{z - w} dz \wedge d\bar{z}\Big).$$

In particular, we have for $f \in C^1(\bar{D}) \cap \mathcal{O}(D)$ that

$$f(w) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)}{z - w} dz.$$

Proof. The proof requires some familiarity with the exterior form calculus, see also Section 2.1

Let $D_\delta = D_\delta(w) \subset D$ and

$$\eta_w(z) := \frac{1}{2\pi i} \cdot \frac{f(z)}{z - w} dz$$

which is defined on $D \backslash D_\delta$. Its exterior derivative is the continuous 2-form on $\overline{D \backslash D_\delta}$ given by

$$d\eta_w(z) = \frac{1}{2\pi i} \cdot \frac{1}{z - w} \bar{\partial} f d\bar{z} \wedge dz.$$

In particular, the integral on the left hand side exists which via Stokes gives

$$\int_{\overline{D \backslash D_\delta}} d\eta = \int_{\partial D} \eta - \int_{\partial D_\delta} \eta. \tag{11}$$

Now parametrise ∂D_δ by $\gamma(t) = w + \delta e^{it}$. then

$$\begin{aligned}\int_{\partial D_\delta} \eta &= \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - w} dz \\ &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(w + \delta e^{it})}{\delta e^{it}} i\delta e^{it} dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(w + \delta e^{it}) dt.\end{aligned}$$

The latter expression converges to $f(w)$ as $\delta \to 0$ (consider $f(w + \delta e^{it})$ as a function in δ and apply Taylor's theorem). On the other hand

$$\Big|\frac{\bar{\partial} f}{z - w} d\bar{z} \wedge dz\Big| = 2\frac{|\bar{\partial} f|}{r} r|dr \wedge dt| \leqslant 2c|dr \wedge dt|$$

where we introduced polar coordinates $x = r\cos t$ and $y = r\sin t$ so that $d\bar{z} \wedge dz = 2irdr \wedge dt$. Hence $d\eta$ is absolutely integrable which implies

$$\int_{\overline{D\backslash D_\delta}} d\eta = \int_{D\backslash D_\delta} d\eta = \int_D d\eta - \int_{D_\delta} d\eta.$$

However, $|\int_{D_\delta} d\eta| \leqslant C\delta^2$ for some constant C which by (11) implies

$$\int_D d\eta = \int_{\partial D} \eta + \lim_{\delta\to 0}\big(\int_{D_\delta} d\eta - \int_{\partial D_\delta} \eta\big) = \int \partial D\eta - f(w),$$

whence the result. (Note that we cannot apply Stokes to $\int_{D_\delta} d\eta$ for $d\eta$ is singular at $w \in D_\delta$.) □

We will now draw several very powerful corollaries.

11. Corollary. *Let $f \in \mathcal{O}(U)$.*

(i) *f is analytic. More precisely, for all $z_0 \in U$ we have $f(z) = \sum_{k=0}^\infty a_k(z - z_0)^k$ for $z \in D_\epsilon(z_0)$ where $\epsilon > 0$ is any real number $< \operatorname{dist}(z_0, \partial U)$.*

(ii) *If U is connected and $\mathcal{Z}(f) = \{z \in U \mid f(z) = 0\}$ has an accumulation point, then $f \equiv 0$ (**"Identity Theorem"**).*

Proof. (i) By Cauchy's theorem for all $w \in D = D_\epsilon(z_0)$ we have

$$\begin{aligned}
f(w) &= \frac{1}{2\pi i}\int_{\partial D}\frac{f(z)}{z-w}dz \\
&= \frac{1}{2\pi i}\int_{\partial D}\frac{f(z)}{(z-z_0)-(w-z_0)}dz \\
&= \frac{1}{2\pi i}\int_{\partial D}\frac{f(z)}{z-z_0}\cdot\frac{1}{1-\frac{w-z_0}{z-z_0}}dz \\
&= \frac{1}{2\pi i}\int_{\partial D}\big(\sum_{k=0}\frac{f(z)}{(z-z_0)^{k+1}}dz\big)(w-z_0)^k \\
&= \sum_{k=0}^\infty\big(\frac{1}{2\pi i}\int_{\partial D}\frac{f(z)}{(z-z_0)^{k+1}}dz\big)(w-z_0)^k.
\end{aligned}$$

Note that all we used is that $\bar{D} \subset U$ and $|w - z_0| < |z - z_0|$ since $w \in D$ and $z \in \partial D$. Hence the computation is valid for any $\epsilon < \operatorname{dist}(x, \partial U)$.

(ii) Let $z_0 \in \mathcal{Z}(f)$ and $f \not\equiv 0$ near z_0. By (i) we can write $f(z) = (z - z_0)^m g(z)$ with $g(z_0) \neq 0$ and $m \geqslant 1$ unless $f \equiv 0$ near z in some open neighbourhood V of z_0. Hence $g(z) \neq 0$ for z sufficently close to z_0, so that $\mathcal{Z}(f) \cap V = \{z_0\}$. So whenever $\mathcal{Z}(f)$ has an accumulation point a, then f must vanish identically near that point. Since a power series around z_0 is identically zero on $D_\epsilon(z_0)$ if its zero set has an accumulation point (see for instance [Pö, 9.40]). This entails that $f \equiv 0$ on U if U is connected. Indeed, let $f \equiv 0$ on V_a near the accumulation point a and let $z \in \mathcal{Z}(f)$. Choose a curve $u : I := [0, 1] \to \mathbb{C}$ from a to z and let $\epsilon < \operatorname{dist}(\operatorname{Im} u, \partial U)$. Then f vanishes on $D_\epsilon(z_0, \epsilon)$. If $w \in D_\epsilon(z_0, \epsilon) \cap \operatorname{Im} u$, then f vanishes on $D_\epsilon(z_0, \epsilon) \cup D_\epsilon(w, \epsilon)$. Since $\operatorname{Im} u$ is compact we find after a finite number of steps that z lies in a disc of radius ϵ on which f vanishes. Hence $\mathcal{Z}(f)$ is open. Since it is closed for f is continuous, $\mathcal{Z}(f) = u$ for U is connected. □

12. Remark.

(i) Because of (i) of the previous corollary one uses the terms "complex analytic" and "holomorphic" interchangeably. Note that in particular, a holomorphic function is automatically C^∞ (which also immediately shows that there are C^1 functions which are not holomorphic).
(ii) If U is connected, then $\mathcal{O}(U)$ is an integral domain.

13. Corollary (mean value inequality). *For all $f \in \mathcal{O}(U)$ we have $|f(w)| \leqslant \max_{z\in\partial D_\epsilon(w)} |f(z)|$ whenever $\partial D_\epsilon(w) \subset U$.*

Proof. From Cauchy's theorem we immediately get that

$$|f(w)| \leqslant \frac{1}{2\pi}\int_{\partial D_\epsilon(w)} |\frac{f(z)}{z-w}|dz = \frac{1}{2\pi}\int_0^{2\pi} f(w+\epsilon e^{it})dt \leqslant \max_{z\in\partial D_\epsilon(w)} |f(z)|.$$

□

14. Corollary (holomorphic functions are open). *Let $f \in \mathcal{O}(U)$ be nonconstant and U connected $\Rightarrow$ f is* **open***, i.e. images of open sets are again open.*

Proof. We proceed in two steps.

Step 4. *Let D be an open disc around $w \in U$ whose closure $\bar{D}$ is contained in U, and such that* $\min_{z\in\partial D} |f(z)| > |f(w)| \Rightarrow$ *there exists $z_0 \in D$ with $f(z_0) = 0$.* If not, there exists an open neighbourhood V of $\bar{D}$ where $f(z) \neq 0$ so that $g := 1/f|_V \in \mathcal{O}(V)$. By assumption, $\max_{z\in\partial D} |g(z)| < |g(w)|$ which is impossible by the mean value inequality.

Step 5. *Conclusion.* Let $z \in D$. We need to show that there exists $\epsilon > 0$ such that $D_\epsilon(f(z)) \subset f(D)$. Since f is not constant, there exists $\delta > 0$ such that $f(z) \notin f(\partial D_\delta(z))$. (Otherwise, there would exist for any $\delta > 0$ an element $z_\delta \in \partial D_\delta$ with $f(z) = f(z_\delta)$ which by the Identity Theorem implies $f \equiv$ *const* on U, a contradiction.) For such a δ we define $2\epsilon := \min_{w\in\partial D_\delta(z)} |f(w) - f(z)| > 0$. Now for all $w \in \partial D_\delta(z)$,

$$v \in D_\epsilon(f(z)) \Rightarrow |f(w) - v| \geqslant |(fw) - f(z)| - |f(z) - v| > 2\epsilon - \epsilon > \epsilon.$$

Hence $\min_{w\in\partial D_\delta(z)} |f(w) - v| > \epsilon > |f(z) - v|$. By the first step this yields a $z_0 \in D_\delta(z)$ such that $f(z_0) - v = 0$, whence $f(z_0) = v \in f(D_\delta(z))$.

□

15. Corollary (maximal principle). *If for $f \in \mathcal{O}(U)$, $z \in U$ is a local maximum of $|f|$, then f must be constant.*

Proof. Let $z \in U$ be such that $|f(z)| \geqslant |f(w)|$ for all $w \in D := D_\epsilon(z) \subset U$. But then $f(D)$ is not open, for if it were, there would exists a disc $D_\delta(f(z)) \subset f(D)$ and thus $f(w) \in D_\delta(f(z))$ with $w \in D$ and $|f(w)| > |f(z)|$. Hence f must be constant. □

16. Corollary (Liouville). *If f is entire, i.e. $f \in \mathcal{O}(C)$ and bounded $\Rightarrow$ $f \equiv const$.*

Proof. Assume that $|f(z)| \leqslant M$ for all $z \in \mathbb{C}$. If $w \in \mathbb{C}$ and $\delta > |w|$ we have from Cauchy that

$$f(w) - f(0) = \frac{1}{2\pi i}\int_{\partial D_\delta(0)} f(z)\Big(\frac{1}{z-w} - \frac{1}{z}\Big)dz = \frac{w}{2\pi i}\int_{\partial D_\delta(0)} \frac{f(z)}{z(z-w)}dz.$$

It follows that $|f(w) - f(0)| \leqslant |w| \max_{|z|=\delta} \frac{|f(z)|}{|z-w|} \leqslant |w| M \max_{|z|=\delta} \frac{1}{|z-w|}$. Now $|z-w| \geqslant \big||z| - |w|\big| = \delta - |w|$ so that finally,

$$|f(w) - f(0)| \leqslant \frac{|w| \cdot M}{\delta - |w|}$$

Since we can choose δ arbitrarily big, $f(w) = f(0)$. □

17. Remark. The previous statement is obviously false for smooth functions as the existence of cut-off functions demonstrates.

18. Holomorphic logarithm. A final aspect of holomorphic functions we want to discuss is the existence of *holomorphic logarithms*. By definition, a holomorphic logarithm of a holomorphic function $f \in \mathcal{O}(U)$ is a holomorphic function $\log g$ which satisfies $\exp(\log g) = g$. To illustrate the problem, consider the identity function $z = re^{i\arg(z)}$ where $\arg(z) \in [0, 2\pi)$ gives the *argument of* z, i.e. its angle for standard polar coordinates (r, φ) of $\mathbb{R}^2 \cong \mathbb{C}$ (with the convention that $\arg(0, y) = 0$ for $y > 0$). If for $z \neq 0$ we define $\log z = \log r + i\arg(z)$, then obviously $\exp(\log z) = re^{i\arg(z)} = z$, and by the local injectivity of exp we know that this is the only solution if we restrict to suitable small domains and ranges of the functions involved. However, we cannot define $\log z$ globally on $\mathbb{C}^*$ for going around the origin once shows that $\arg(z)$ is not continuous – the limit $\lim_{\varphi \to 2\pi-} \arg(re^{i\varphi}) = 0$, not φ. We therefore can only take holomorpic logarithms of holomorphic functions $f : U \to \mathbb{C}$ defined on *simply-connected* domains such that $f(z) \neq 0$ for all $z \in U$, for instance by taking a small disc around any z where f is defined with $f(z) \neq 0$. Once we have defined holomorphic logarithms we can also define *holomorphic roots*, for instance $\sqrt{f} = \exp(\frac{1}{2}\log f)$ etc.

Next we will consider singularities of holomorphic functions.

19. Definition (singularities of holomorphic functions). If $f \in \mathcal{O}(U \backslash \{z_0\})$, then z_0 is called a **singularity** of f. If f is bounded near z_0, then the singularity is called **removable**. If there exists $m \in \mathbb{N}$ such that $(z - z_0)^m \cdot f(z)$ is bounded near z_0, and if m is taken to be minimal with this property, then z_0 is called a **pôle of order** m **of** f. In all other cases, f is called an **essential singularity**.

We investigate these types of singularities next. First we characterise removable singularities.

20. Theorem (Riemann's removable singularities theorem). *Let $f \in \mathcal{O}(U \backslash \{z_0\})$, and $z_0 \in U$ be a singularity. Are equivalent:*

(i) z_0 *is removable;*
(ii) $\lim_{z \to \infty}(z - z_0)f(z) = 0$*;*
(iii) f *extends to a continuous function on all of* U*;*
(iv) f *extends to a holomorphic function on all of* U*.*

21. Remark. Note that the extension, if it exists, is uniquely determined by the Identity Theorem A.20.

Proof. Up to a translation we may assume without loss of generality that $z_0 = 0$ and put $U^\times = U\backslash\{0\}$. It is clear that $(iv) \Rightarrow (iii) \Rightarrow (ii) \Rightarrow (i)$ and $(i) \Rightarrow (ii) \Rightarrow (iii)$. We are now going to prove the nontrivial direction.

(ii)⇒(iv): Let $g(z) := zf(z)$ on U^times and $g(0) := 0$. By assumption, g is continuous in 0, hence on all of U. Moreover, $h(z) := zg(z) \in C^0(U) \cap \mathcal{O}(U^\times)$, and $h(0) = 0$. It follows that

$$\frac{h(z) - h(0)}{z} = \frac{h(z)}{z} \to 0$$

so that h is holomorphic with $h'(0) = 0$ and hence analytic near 0, i.e.

$$h(z) = \sum_{i=1}^{\infty} a_i z^i = z^2 \sum_{i=0}^{\infty} a_{i+2} z^i = z^2 f(z).$$

It follows that $\sum_{i=0}^{\infty} a_{i+2} z^i$ is the desired holomorphic extension of f. □

Next we consider the case of poles.

22. Proposition (characterisation of poles). *Let $f \in \mathcal{O}(U\backslash\{z_0\})$, $z_0 \in U$ be a singularity, and n be an integer $\geqslant 1$. Are equivalent:*

(i) *f has a pole of order n at z_0;*

(ii) *there is a function $g \in \mathcal{O}(U)$ with $g(z_0) \neq 0$, and such that*

$$f(z) = \frac{g(z)}{(z - z_0)^n} \qquad z \in U\backslash\{z_0\};$$

(iii) *there is an open neighbourhood V of z_0 in U and $h \in \mathcal{O}(V)$ without zeroes on $V\backslash\{z_0\}$, with a zero in z_0 of order n, and such that $f = 1/h$ on $V\backslash\{z_0\}$;*

(iv) *there is a neighbourhood V of z_0 lying in U and positive constants c, C such that for all $z \in V\backslash\{z_0\}$*

$$c \leqslant |f(z)||z - z_0|^n \leqslant C.$$

In particular, the function $\mathcal{O}(U\backslash\{z_0\})$ has a pole at z_0 if and only if $\lim_{z\to z_0} f(z) = \infty$.

Proof. Let $U^\times = U\backslash\{z_0\}$. Again we assume for simplicity that $z_0 = 0$.

(i)→(ii) Since $z^n f \in \mathcal{O}(U^\times)$ is bounded near z_0 we can remove the singularity and obtain a holomorphic function $g(z) = z^n f(z) \in \mathcal{O}(U)$. If $g(0) = 0$, then $g(z) = z\tilde{g}(z)$ with $\tilde{g} \in \mathcal{O}(U)$ which implied that $\tilde{g}(z) = z^{n-1} f(z)$ on $U^\times$ so that $z^{n-1} f(z)$ would be bounded near z_0 – a contradiction to the minimality of the pole order n.

(ii)⇒(iii) Since $g(0) \neq 0$, g does not vanish on a neighbourhood of 0. Thus $h(z) = z^m/g(z)$ yields the desired function.

(iii)⇒(iv) On a suitable neighbourhood V of 0, $h(z) = z^m \tilde{h}(z)$ for $\tilde{h} \in \mathcal{O}(V)$ with $\tilde{h}(0) \neq 0$. Hence there exist constants c and C such that $c < |\tilde{h}(z)| < C$. Since $|f(z)|^{-1} = |z|^m |\tilde{h}(z)|$, the claim follows.

(iv)⇒(i) By assumption, $z^m f$ is bounded. Furthermore, $|z|^{m-1} f(z)| \geqslant C/z$ which shows that $z^{m-1} f$ is not bounded near $z_0 = 0$, whence m is minimal and equals the order of the pole. □

23. Proposition (development into a Laurent series with finite principal part). *Let $f \in \mathcal{O}(U\backslash\{z_0\})$, and let z_0 be a pole of order n. Then there exist complex numbers $b_1, \ldots, b_n$ with $b_n \neq 0$, and a holomorphic function $\tilde{f} \in \mathcal{O}(U)$, such that*

$$f(z) = \frac{b_m}{(z - z_0)^m} + \frac{b_m}{(z - z_0)^m} + \ldots + \frac{b_m}{(z - z_0)^m} + \tilde{f}(z), \quad z \in U\backslash\{z_0\}.$$

The numbers b_i and the function $\tilde{f}$ are uniquely determined. Conversely, any function of this form has a pole of order n at z_0.

Proof. By (ii) of Proposition A.22, $f(z) = g(z)/(z - z_0)^m$ on $U\backslash\{z_0\}$, where $g \in \mathcal{O}(U)$. Hence g is analytic around z_0, and developping into a series around z_0 yields the Laurent series of f. The remaining assertions are clear. □

Finally, we characterise essential singularities.

24. Theorem (Casorati-Weierstrass). *Let $f \in \mathcal{O}(U\backslash\{z_0\})$, and $z_0 \in U$ be a singularit. Are equivalent:*

(i) *z_0 is an essential singularity;*
(ii) *for every neighbourhood $V \subset U$ of z_0, the image $f(V\backslash\{z_0\})$ is dense in $\mathbb{C}$;*
(iii) *there exists a sequence z_n in $U\backslash\{z_0\}$ with $\lim z_n = z_0$, and such that the image sequence $f(z_n)$ has no limit in $\mathbb{C} \cup \{\infty\}$.*

Proof. Let as above $U^\times := U\backslash\{z_0\}$. We prove the nontrivial assertion (i)⇒(ii) by contradiction. Assume therefore that there exists no such neighbourhood V of z_0 in U, that is, there exists a disc $D := D_\epsilon(a) \subset \mathbb{C}$ such that $D \cap f(U^\times) = \varnothing$. In particular, $|f(z) - a)| > r$ so that the function $g(z) := 1/(f(z) - a)$ is holomorphic and bounded on $U^\times$. Therefore, the singularity of g in z_0 is removable. Thus either $f(z) = a + 1/g(z)$ has a removable singularity if $g(z_0) \neq 0$, or it has a pole if $g(z) = 0$. However, we assumed z_0 to be essential, contradiction! □

25. Example. Using Proposition A.22 it is easy to see that $\exp(1/z)$ has an essential singularity at $z = 0$.

It is actually desirable to allow poles which gives rise to the following definition.

26. Definition (meromorphic function). A function f is called **meromorphic** on U if there is a discrete subset $\mathcal{P}(f) \subset U$, the **pole set**, such that $U\backslash\mathcal{P}(f)$ is holomorphic in $U\backslash\mathcal{P}(f)$ and has a pole at any $z_0 \in \mathcal{P}(f)$. In particular, a meromorphic function on U is holomorphic if and only if $\mathcal{P}(f) = \varnothing$. We often set $f(z) := \infty$ for $z \in \mathcal{P}(f)$ and consider a meromorphic function as a map $f : U \to \mathbb{C} \cup \{\infty\} = \mathbb{P}^1$.

27. Examples.

(i) Let g, $h \in \mathbb{C}[z]$ be two polynomials. Then $f = g/h$ is a meromorphic function whose pole set is given by the zero set of h. We call f also a **rational function**.
(ii) The cotangent function $\cot z = \cos z/\sin z$ is meromorphic, but not rational. Indeed, it has pole set $P(\cot z) = \mathcal{Z}(\sin z) = \pi\mathbb{Z}$ which is countably infinite.

To discuss further existence theorems of holomorphic and meromorphic functions we first introduce the **order function**. If $f \in \mathcal{O}(U)$ is not identically zero, then there is a minimal natural number $\nu_z(f)$ such that $f(z) = f'(z) = \ldots = f^{(m-1)}(z) = 0$ and $f^m \neq 0$. If $f \equiv 0$ we put $\nu_z(0) = \infty$. We extend this to the case of meromorphic functions by letting $\nu_z(f) = -m$ if z is a pole of f of order m. Thus if f and g are meromorphic on U,

(i) f is holomorphic at $z \in U \Leftrightarrow \nu_z(f) \leqslant 0$. In this case, $f(z) = 0 \Leftrightarrow \nu_z(f) > 0$.
(ii) $\nu_z(fg) = \nu_z(f) + \nu_z(g)$.
(iii) $\nu_z(f+g) \geqslant \min\{\nu_z(f), \nu_z(g)\}$ with equality whenever $\nu_z(f) \neq \nu_z(g)$.

We have two classical existence results.

28. Theorem (Weierstrass). *Let $U \subset \mathbb{C}$ be open, and let $S \subset U$ be a discrete subset, and $o : S \to \mathbb{N}_{>0}$ be a given function. Then there exists $f \in \mathcal{O}(U)$ with $\nu_a(f) = o(a)$ for all $a \in S$. Put differently, we can always construct holomorphic functions with given zeroes of any order.*

29. Theorem (Mittag-Leffler). *Let $U \subset \mathbb{C}$ be open, and let $S \subset U$ be a discrete subset. Then there exists $f \in \mathcal{M}(U)$ with $\mathcal{P}(f) = S$ and such that the Laurent series of f has any given prinicipal part at that point.*

We give (at least partial) proofs in Section 2.1, see also [Fo, Theorems 26.3 and 26.7]. Weierstrass' theorem is actually a corollary to the Weierstrass Factorisation Theorem.

It is clear that we can add and multiply meromorphic function over U in a natural way, namely by adding or multiplying the holomorphic functions outside the pole set and by computing the new pole set of the result; $\mathcal{P}(f \pm g), \mathcal{P}(f \cdot g) \subset \mathcal{P}(f) \cup \mathcal{P}(g)$. Moreover, we can also divide by meromorphic functions. This turns $\mathcal{M}(U)$ into a $\mathbb{C}$-algebra. If $f \in \mathcal{M}(U)$ its zero set $\mathcal{Z}(f)$ is discrete. Now on $U \backslash (\mathcal{P}(f) \cup \mathcal{Z}(f))$, f is holomorphic without zeroes, so that $1/f$ is defined and also holomorphic. Moreover, $z_0 \in \mathcal{Z}(f)$ becomes a pole of $1/f$ of same order as the zero of f. Moreover, since for $z_0 \in \mathcal{P}(f)$, $(z_0 - z)^m f(z)$ is bounded for some m, $\lim_{z \to z_0} 1/f(z) = 0$ so that $1/f$ has a removable singularity in z_0. It follows that $\mathcal{M}(U)$ is actually a field. In fact we have the

30. Proposition. *Let $U \subset \mathbb{C}$ be connected. Then $\mathcal{M}(U) = \operatorname{Quot} \mathcal{O}(U)$, that is, $\mathcal{M}(U)$ is the quotient field of the integral domain $\mathcal{O}(U)$.*

Proof. It is clear that $\operatorname{Quot} \mathcal{O}(U) \subset \mathcal{M}(U)$. Conversely, let $f \in \mathcal{M}(U)$. By the Weierstrass Factorisation Theorem A.28 we can find a holomorphic function $h \in \mathcal{O}(U)$ whose zero set is precisely $\mathcal{P}(f)$ and such that $\nu_z(g) = -\nu_z(f)$. In particular, if $g = h \cdot f$, we have $\nu_z(g) = \nu_z(g) + \nu(f) \leqslant 0$ so that $g \in \mathcal{O}(U)$. Hence $f = g/h \in \operatorname{Quot} \mathcal{O}(U)$. □

Finally, we mention how zeroes and poles can be detected by integration. Let $f \in \mathcal{M}(U)$ be a meromorphic function, and let D be any disc in U containing finitely many poles and zeroes of f. We let $Z_D(f) = \sum_{p \in D,\, o_p(f) > 0} o_p(f)$ and $P_D(f) = \sum_{p \in D,\, o_p(f) < 0} o_p(f)$ the total order of zeroes and poles of f in D, that is, the number of zeroes and poles counted with multiplicity. As a corollary to the so-called residue theorem [Pö, Section 9.8] we obtain

31. Rouché's formula. *If $f \in \mathcal{M}(U)$ and D is any disc in U containing finitely many zeroes and poles of f $\Rightarrow$*

$$\frac{1}{2\pi i}\int_{\partial D}\frac{f'(z)}{f(z)}dz = Z_D(f) - P_D(f).$$

Appendix B. Covering spaces

In this appendix we summarise, mostly without proof, the basics of homotopy theory and covering spaces. For details see [Fo, Chapter 1, §3-§5] or [Fu, Chapter 11-13]. Here, X will denote a topological space.

The fundamental group. A **(parametrised) curve** is a continuous map $u : I := [0,1] \subset \mathbb{R} \to X$. $u(0) \in X$ is called the **initial point** and $u(1) \in X$ the **final point**. We say that u joins $u(0)$ to $u(1)$. A **reparametrisation** is a continuous map $\varphi : I \to I$ such that $\varphi(0) = 0$ and $\varphi(1) = 1$. The curve $u \circ \varphi$ is called a reparametrisation of u. They have, of course, the same image, so be careful to distinguish between the locus of points defined by $\operatorname{im} u$ and the map u itself.

1. Definition (homotopy of curves). Two curves u and $v : I \to X$ from $a \in X$ to $b \in X$ are called **homotopic**, and we write $u \sim v$, if there exists a **homotopy** or **deformation**, i.e. a continuous map $H : I \times I \to X$ such that for all $s, t \in I$ we have

- $H(t, 0) = u(t)$;
- $H(t, 1) = v(t)$;
- $H(0, s) = a$ and $H(1, s) = b$;

see also Figure B.7.

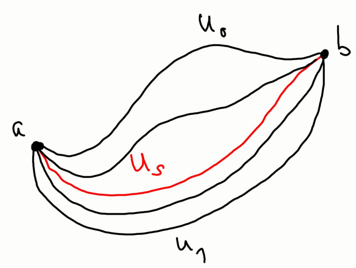

Figure 7. Homotopy of the curves u_0 and u_1

2. Lemma [Fo, Theorem 3.2 and Lemma 3.3]**.** *Homotopy defines an equivalence relation on the set of curves between two given points a and b in X. In particular, u is equivalent to any of its reparametrisations. We write $[u]$ for this equivalence class.*

Next we aim to define a group structure on the set of **closed curves**, i.e. curves $u : I \to X$ with $u(0) = u(1)$.

3. Definition (product of curves, inverse and constant curves). Let $u : I \to X$ be a curve from a to b.

(i) If $v : I \to X$ is a curve from b to c, then the **product curve** $u * v : I \to X$ is defined by

$$(u * v)(t) := \begin{cases} u(2t) & \text{for } 0 \leqslant t \leqslant 1/2 \\ v(2t-1) & \text{for } 1/2 \leqslant t \leqslant 1 \end{cases}$$

(ii) The **inverse curve** $u^- : I \to X$ is defined by

$$u^-(t) := u(1-t) \text{ for every } t \in I.$$

(iii) The **constant curve at** a is the curve $u_a : I \to X$ defined by $u_0(t) = a$ for all $t \in I$.

(iv) The curve $u : I \to X$ is **null-homotopic** if it is homotopic to the constant curve at $u(0)$.

Note that taking products or the inverse is compatible with the relation of homotopy, that is,

$$u_1 \sim v_1, \quad u_2 \sim v_2 \Rightarrow u_1 * u_1 \sim v_1 * v_2 \text{ and } u_1^- \sim u_2^-.$$

We are now prepared for the

4. Theorem and Definition [Fo, Theorem and Definition 3.8]**.** *Let X be a topological space and $a \in X$ is a point. The set $\pi_1(X, a)$ of homotopy classes of closed curves at a forms a group under the product and inverse as defined in Definition B.3. This group is called the* **fundamental group of X with base point** *a.*

A priori, the fundamental group depends on the base point. However, if a and b are two points in X which are joined by a curve u, then we can define a group isomorphism

$$\Gamma_u : \pi_1(X, a) \to \pi_1(X, b), \quad [v] \mapsto [u^- * v * u]$$

This motivates the following

5. Definition ((locally) arcwise connected). X is **arcwise connected** if any two point can be joined by a curve. An arcwise connected space is connected, the converse is true if X is in addition **locally arcwise connected**, that is, every point has a neighbourhood basis of arcwise connected sets. Since for an arcwise connected space, the fundamental group is determined up to group isomorphism we usually speak of *the* fundamental group of the arcwise connected space and write $\pi_1(X)$.

6. Example. A Riemann surface is always locally arcwise connected for it has neighbourhoods homeomorphic to $\mathbb{R}^2$. Since they are connected, they are automatically arcwise connected.

7. Definition (simply connected). *An arcwise connected space X is called simply connected if its fundamental group $\pi_1(X)$ is trivial.*

8. Examples.

(i) A subset $X \subset \mathbb{R}^n$ is **star-shaped with respect to** $a \in X$ if for every point $b \in X$, the line segment $ta + (1-t)b$, $t \in I$, is contained in X, see Figure B.8. In particular, X is (arcwise) connected. If X is star-shaped, then X is simply connected. Indeed, if u is a closed curve at a, $H : I \times I \to X$, $H(t,s) = sa + (1-s)u(t)$ is a homotopy from u to u_a. In particular, $\mathbb{C} \cong \mathbb{R}^2$ or any slitted plane such as $\mathbb{C}\backslash\mathbb{R} > 0$ are simply connected.

(ii) The Riemann sphere is simply connected. Indeed, let u be a closed curve, say at $\infty = [0 : 1]$. If u is contained in $U_1 \subset \mathbb{P}^1$, then we can deform u into u_∞ for U_1 is star-shaped. If not, we factorise $u = u_1 * \ldots * u_r$ by subdividing I into smaller intervals and reparametrising the resulting curves, so that the u_{2k+1} lie entirely in U_1 and u_{2k} lie entirely in U_0. Now we can slightly deform the curves u_{2k} into $\tilde{u}_{2k}$ so that they lie in $U_0\backslash\{[1 : 0]\} \subset U_1$. Hence u can be deformed to $u' = u_1 * \tilde{u}_2 * \ldots$ which lies in U_1 and is therefore null-homotopic.

(iii) The torus is not simply connected. In fact, $\pi(T_\Lambda) \cong \mathbb{Z}[\alpha] \oplus \mathbb{Z}[\beta]$, where the curve $n[\alpha]$ means winding round n times α, and similarly for $[\beta]$, see Figure B.9 and Appendix C (recall that T_Λ is homeomomorphic to $S^1 \times S^1$, cf. Remark 1.7).

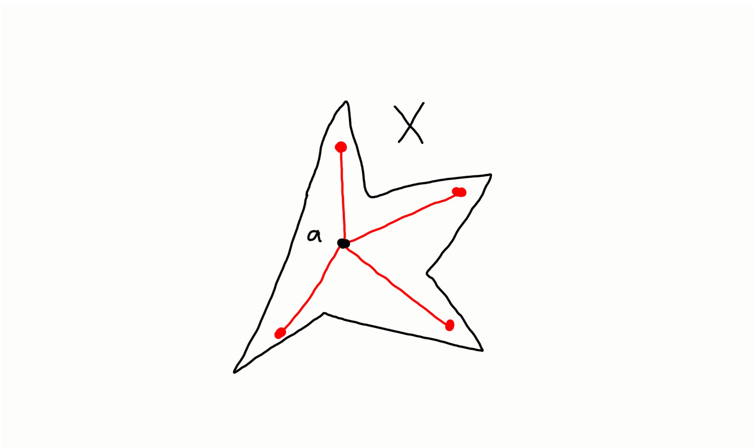

FIGURE 8. A starshaped domain in the plane

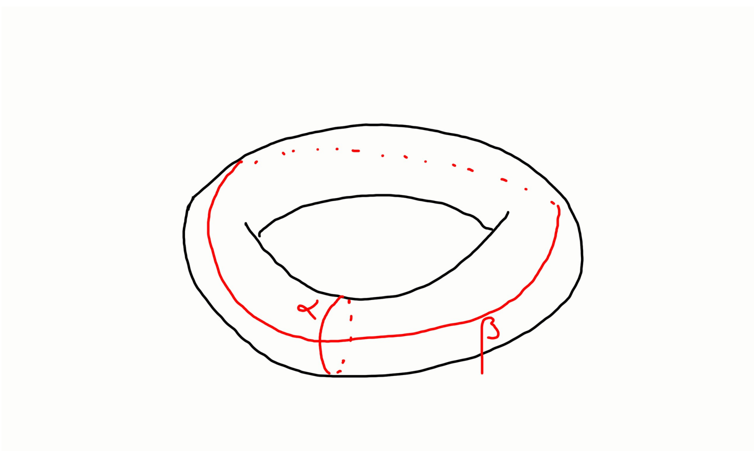

FIGURE 9. The generators α and β of the fundamental group of the torus

The previous example immediately implies that T_Λ cannot be homoemorphic to $\mathbb{P}^1$ for the fundamental group is a *topological invariant.*

9. Definition (pushforward map). Let $f : X \to Y$ be a continuous map between two topological spaces. Then we define the **pushforward of** f by

$$f_* : \pi_1(X, a) \to \pi_1(Y, f(a)), \quad f_*[u] = [f \circ u].$$

This is indeed well-defined and satisfies $(f \circ g)_* = f_* \circ g_*$ (see [Fo, 3.15]).

10. Application: The fundamental group is a topological invariant. It follows that if f is a homeomorphism, then f_* and f_*^{-1} are defined, and $\mathrm{Id}_{\pi_1(X,a)} = (f^{-1} \circ f)_* = f_*^{-1} \circ f_*$, and similarly for $\pi_1(Y, f(a))$. Hence f_* and f_*^{-1} are group isomorphisms which are inverse to each other. In particular, the two fundamental groups must be isomorphic.

Lifting of curves and covering maps. One is often confronted with the following *lifting problem*: Given two continuous maps $p : X \to Y$ and $f : Z \to Y$, when does there exists a (unique) map $g : Z \to X$ such that $p \circ g = f$?

$$\begin{tikzcd} & X \arrow[d, "p"] \\ Z \arrow[ur, dashed, "g"] \arrow[r, "f"'] & Y \end{tikzcd}$$

This question leads to an important subclass of homeomorphisms, namely *covering maps.* Since we are mainly interested in applications to Riemannian surfaces **we will assume that** X **and** Y **are Hausdorff spaces** which slightly simplifies the presentation. While existence of lifting is a subtle issue, uniqueness holds if p is a local homeomorphism:

11. Proposition (uniqueness of liftings) [Fo, Theorem 4.8]**.** *Let $p : X \to Y$ be a local homeomorphism, and let $f : Z \to Y$ be a continuous mapping from some connected topological space Z. If g_1 and g_2 are two liftings of f in the sense above which agree in one point, i.e. $g_1(z) = g_2(z)$ for one $z \in Z$, then $g_1 = g_2$.*

To discuss existence we first consider a special case.

12. Definition (curve lifting property). A continuous map $p : X \to Y$ is said to have the **curve lifting property** if for every curve $u : I \to Y$ and every point $\tilde{a} \in X$ there exists a lift $\tilde{u} : I \to X$ such that $\tilde{u}(0) = \tilde{a}$ and $p \circ \tilde{u} = u$. (By Proposition B.11 the lift is uniquely determined by $\tilde{a}$.)

Next we give a criterion when homotopy is preserved under lifts.

13. Proposition [Fo, Theorem 4.10]**.** *Let $p : X \to Y$ be a local homeomorphism. Let $H : I \times I \to Y$ be a (continuous) homotopy with $H(0, s) = a$ and $H(1, s) = b$, and let $\tilde{a} \in X$ be such that $p(\tilde{a}) = a$. If all curves $u_s(t) := H(t, s) : I \to Y$ can be lifted to X with initial point $\hat{a}$, then $\tilde{A}(t, s) := \tilde{u}_s : I \times I \to X$ defines a homotopy between $\tilde{u}_0$ and $\tilde{u}_1$. In particular, $\tilde{u}_s(1)$ is constant so that $\tilde{u}_0$ and $\tilde{u}_1$ have the same endpoint.*

14. Corollary (constant number of sheets)[Fo, Theorem 4.16]**.** *If $p : X \to Y$ has the curve lifting property and Y is arcwise connected, then for any two points*

a and $b \in Y$, the fibres $p^{-1}(a)$ and $p^{-1}(b)$ have the same cardinality. In particular, p is surjective.

The curve lifting property gives a handy criterion for the existence of lifts of general maps:

15. Theorem [Fo, Theorem 4.17]**.** *Let $p : X \to Y$ be a local homeomorphism which has the curve lifting property. If Z is a connected and simply connected space, and $f : Z \to Y$ a continuous map, then there exists a lift of f which is uniquely determined by any choice of points $f(a) = b$, $a \in Z$ and $b \in X$.*

It remains to find conditions which ensure the curve lifting property.

16. Definition (covering map). A local homeomorphism $p : X \to Y$ is called a **covering map** if every point $a \in Y$ has an open neighbourhood U such that its preimage $p^{-1}(U)$ can be represented as

$$p^{-1}(U) = \bigcup_{i \in I} U_i,$$

where the U_i, $i \in I$, are disjoint open subsets of X, and all the maps $p|_{U_i} \to U$ are homeomorphisms. Any open neighbourhood U with this property is said to be **special**. A **morphism of covering maps** $p : X \to Y$ and $q : Z \to Y$ is a continuous map $f : X \to Z$ such that the diagramm

$$\begin{tikzcd} & X \arrow[dl, dashed, "f"'] \arrow[d, "p"] \\ Z \arrow[r, "q"] & Y. \end{tikzcd}$$

commutes.

17. Examples.

(i) Consider the holomorphic mapping $p_k : \mathbb{C}^* \to \mathbb{C}^*$, $p_k(z) = z^k$. Since the k-th roots of unity form a discrete finite subset of $\mathbb{C}^*$ it is clear that p_k is a covering map. If we holomorphically extend p_k to a map $\mathbb{C} \to \mathbb{C}$ by sending 0 to itself, then 0 is a ramification point (considered as a point in the domain) as well as a branch point (considered as a point in the range). Its ramification index is $k - 1$ and its multiplicity is k.

(ii) The holomorphic exponential $\exp : \mathbb{C} \to \mathbb{C}^*$, $\exp(z) = \sum_{k=0} z^k/k!$ is an unbranched holomorphic covering map. Indeed, exp is injective on every subset $U \subset \mathbb{C}$ such that no two points z_0 and z_1 differ by an integer multiple of $2\pi i$.

(iii) In a similar vein, the projection map $\pi_\Lambda : \mathbb{C} \to T_\Lambda$ is an unbranched holomorphic covering map.

18. Remark. Let $D = \{z \in \mathbb{C} \mid |z| < 1\}$ be the open unit disk in the complex plane, and let $p : D \to \mathbb{C}$ be the canonical injection. Then p is a local homeomorphism, but not a covering map for any point $z \in \mathbb{C}$ with $|z| = 1$ has no special neighbourhood.

19. Proposition (local homeomorphisms and covering maps) [Fo, Theorems 4.14 and 4.22]**.** *Every covering map $p : X \to Y$ has the curve lifting property. Conversely, if Y has a basis of simply connected open sets (e.g. Y is a Riemann surface), then a local homeomorphism which has the curve lifting property is a covering map.*

20. Application: Logarithm of a holomorphic function. Let X be a simply connected Riemann surface and $f : X \to \mathbb{C}$ a nowhere vanishing holomorphic function. A **logarithm of** f is a function $g : X \to \mathbb{C}$ such that $\exp(g) = f$ for the holomorphic exponential $\exp : \mathbb{C} \to \mathbb{C}^*$. This can be expressed in terms of liftings:

$$\begin{tikzcd} & \mathbb{C} \arrow[d, "\exp"] \\ X \arrow[r, "f"] \arrow[ur, dashed, "g"] & \mathbb{C}^*. \end{tikzcd}$$

By Proposition B.15 a lift g exists which is clearly holomorphic since exp is a covering map. Moreover, it is uniquely determined by points $x \in X$ and $z \in \mathbb{C}$ such that $f(x) = \exp(z)$, namely $g(x) = z$.

Another condition which ensure that a local homeomorphism is a covering map is *properness.*

21. Definition (proper map). A topological space is **locally compact** if every point has a compact neighbourhood. A continuous map between two locally compact spaces is **proper** if the preimage of any compact set is again compact. In particular, any proper map is closed, for in a locally compact space a set is closed if and only if the intersection with every compact set is compact.

22. Proposition. *Let $p : X \to Y$ be a local homeomorphism. Then p is proper $\Leftrightarrow$ p is a covering map with finite fibres.*

Proof. $\Rightarrow$) See [Fo, Theorem 4.22].

$\Leftarrow$) We briefly sketch the converse. A closed continuous surjective map $p : X \to Y$ is called *perfect* if it has compact fibre. Any such map is proper (cf. [Mu, Exercise 12 §26]) so we only need to show that a covering map with finite fibres is closed. Let $S \subset X$ be closed and consider $f(S) \subset Y$. In general, a set A is closed if and only if there is an open covering $\{U_i\}_{i \in I}$ of the total space such that $U_i \cap A$ is closed in A. Here, we apply this to $f(S)$ with the covering of Y provided by the special neighbourhoods U in Definition B.16. Since a finite union of closed sets is again closed, the finiteness of the fibres implies that $p^{-1}(U)$ is a finite union of open sets, whence the result. $\square$

23. Example. For covering maps with infinite fibres this is false as the example $\exp : \mathbb{R} \to S^1$ with $S = \{n + 1/n \mid n \in \mathbb{N},\, n \geqslant 1\}$ shows. Indeed, S is closed while $\exp(S)$ accumulates at $1 \notin \exp(S)$, hence the exp is not closed. (Of course, one sees directly that it is not proper).

Universal coverings. In this section we will construct a canonical covering for a wide class of topological spaces – the so-called *universal covering.*

24. Definition (universal covering). Let $p : X \to Y$ be a covering map between connected topological spaces (e.g. Riemann surfaces). The p is called the **universal covering map** if it satisfies the following *universal property*: For every covering map $f : Z \to Y$ between connected spaces and any choice of points $a \in Z$ and $b \in X$ such that $p(b) = f(a)$, there exists a unique covering map morphism $g : X \to Z$ with $g(b) = a$.

It follows from the universal property that such a universal covering space, if it exists, must be unique up to unique isomorphism, cf. Definition E.4 and Remark E.5. From Theorem B.15 we see that if $p : X \to Y$ is a covering map between with X connected and simply-connected, p *must* be the universal covering (note that if $p : Z \to Y$ is a covering map and Y Hausdorff, then so is Z). The central theorem in the theory of covering spaces is this.

25. Theorem (existence of the universal covering space). *Let X be a connected space with a basis of simply-connected open sets (e.g. a Riemann surface) $\Rightarrow$ There exists a connected, simply connected space $\tilde{X}$ and a covering map $p : \tilde{X} \to X$. $\tilde{X}$ is called the* **universal covering space of** *X.*

Proof. Fix a(n arbitrary) point $x \in X$. Then one constructs $\tilde{X}$ as the set of homotopy classes of curves in X with initial point x where the homotopy is defined as in Definition B.1 except that the final point $u_s(1)$ is not fixed. For details, see [Fu, Theorem 13.20] or [Fu, Theorem 5.3]. □

26. Remark. Our assumptions are not the most general, cf. [Fu, Theorem 13.20]

27. Definition (Deck transformation and Galois coverings). Let $p : X \to Y$ be a covering map. A **Deck transformation** $f : X \to X$ is a fibre-preserving homeomorphism, i.e. $p \circ f = p$. Composition of Deck transformations gives a group which we denote by $\mathrm{Deck}(X/Y)$. Since Deck transformations are fibre preserving, $\mathrm{Deck}(X/Y)$ acts on each fibre. If it acts transitively, then the covering is called a **Galois covering**.

28. Remark. Note that under the assumptions of Proposition B.11, the action of $\mathrm{Deck}(X/Y)$ is necessarily free.

29. Example. The covering map $p : \mathbb{C}^* \to \mathbb{C}^*$, $p(z) = z^k$, is Galois, since the group of Deck transformations are just the k-th roots of unity.

30. Theorem (Deck transformations and the fundamental group). *Let X be connected and $p : \tilde{X} \to X$ the universal covering. Then p is Galois and $\mathrm{Deck}(\tilde{X}/X)$ is isomorphic with the fundamental group $\pi_1(X)$ of X.*

Proof. Let $\tilde{x} \in \tilde{X}$ such that $p(\tilde{x}) = x$, and let v be a representative of the unique homotopy class of curves in $\tilde{X}$ from $\tilde{x}$ to $f(\tilde{x})$. The assignement $\mathrm{Deck}(\tilde{X}/X) \to \pi_1(X, x)$ which maps f to the closed curve $f \circ v_{\tilde{y}} \in \pi_1(X, x)$ is an isomorphism (see [Fo, Theorem 5.6]). □

31. Remark. It follows that the universal covering of a manifold X is in fact a $\pi_1(X)$-prinicipal fibre bundle, cf. for instance [KoNo, Chapter I.5].

More generally, we have the following relationship between subgroups of $\pi_1(X, x)$ and covering maps.

32. Proposition [Fu, Proposition 13.23]**.** *Let Y be a connected topological space with a basis of simply connected open sets $\Rightarrow$*

(i) *for every subgroup H of $\pi_1(Y,b)$ there exists a connected space Y_H and a covering map $p_H : Y_H \to Y$, and a base point $a \in X_H$, such that $p_{H*}\pi_1(Y_H,a) = H$. Any other such covering is canonically isomorphic to Y_H. In particular, the functor π_1 which associates with a connected space the isomorphy class of its fundamental group yields a bijection between subgroups of $\pi_1(Y)$ and (isomorphism classes of) connected covering spaces of Y.*

(ii) *for every subgroup K containing H, there exists a unique covering map $Y_H \to Y_K$ which is compatible with the projections. If H is a normal subgroup of K, then Y_H is a K/H-principal fibre bundle over Y_K, and* $\mathrm{Deck}(Y_H/Y_K) = K/H$.

33. Remark.

(i) In particular, the universal covering $\tilde{Y}$ is just $Y_{\{e\}}$. Note that if $p : X \to Y$ is a covering map between connected and locally path-connected topological spaces, and $p(a) = b$, $a \in X$, then $p_* : \pi_1(X,a) \to \pi_1(Y,b)$ is an injection so that we can always regard the fundamental group of X as a subgroup of $\pi_1(Y,b)$ (cf. [Fu, Proposition 13.1]). In this way, the theory of covering spaces of Y matches the internal group structure of $\pi_1(Y)$.

(ii) It is easy to see that if $\pi : X \to Y$ is a covering and Y is Hausdorff, then so is X. Hence if Y is a Riemann surface, then there exists a unique holomorphic structure on X such that π becomes holomorphic (cf. Proposition 1.40).

We summarise the situation for a Riemann surface in Figure B.10. The red colour designates a normal subgroup.

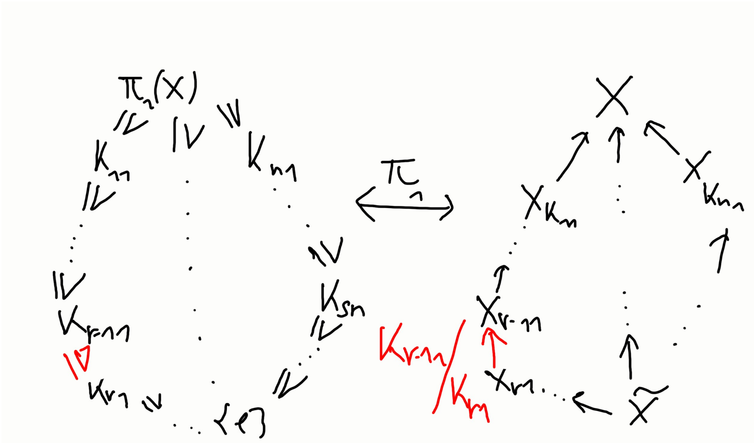

Figure 10. The correspondence between subgroups and covering spaces.

Appendix C. Topology of surfaces

We recall some elements of the topology of (compact) surfaces and discuss in particular some topological invariants and their relationship. Since a lot of concepts are visually clear but quite lengthy to formalise correctly we will mainly appeal to pictures rather than strict definitions. A good reference is [Fu, Part IX] or [Ki]. In this section Σ will denote the toplogical space underlying a compact Riemann surface.

1. Theorem and Definition (genus) [Ki, 4.14]**.** *Σ is homeomorphic to a sphere with g handles (see Figure C.11). We call $g = g(\Sigma)$ the* **genus** *of Σ. It determines Σ up to homeomorphism (and in fact up to diffeomorphism, that is, there is only one uniquely determined differentiable structure on Σ).*

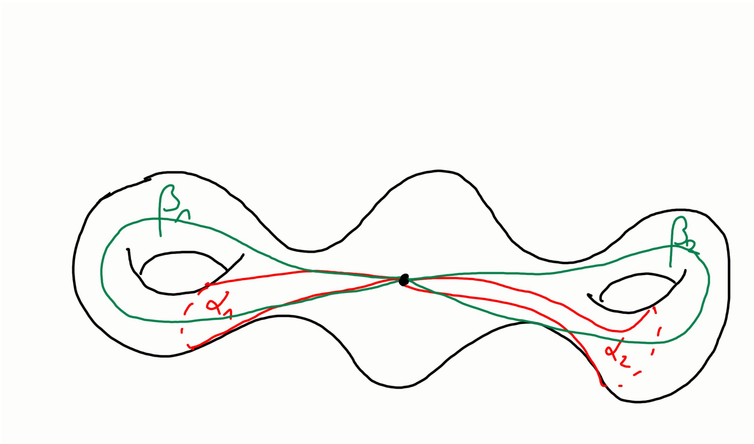

FIGURE 11. The 2-sphere with two handles attached and curves α_i, β_i

The genus is thus the basic topological invariant which so strong that it classifies surfaces. On the other hand, we have already encountered a basic topological invariant of any topological space – its *fundamental group*. In the case of a surface Σ we can compute $\pi_1(\Sigma)$ explicitely. Let F_{2g} be the free group in $2g$ generators $a_1, \ldots, a_g$, $b_1, \ldots, b_g$, where g is the genus of Σ. Further, let N_g be the smallest normal subgroup of F_{2g} containing the element

$$c_g := a_1 \cdot b_1 \cdot a_1^{-1} \cdot b_1^{-1} \cdot \ldots \cdot a_g \cdot b_g \cdot a_g^{-1} \cdot b_g^{-1},$$

i.e. $N_g = \{u \cdot c_g \cdot u^{-1} \mid u \in F_{2g}\}$. Any surface has curves $\alpha_1, \ldots, \alpha_g$ and $\beta_1, \ldots, \beta_g$ inducing distinct homotopy classes also denoted by α_i and β_i (see Figure C.11 for the case $g = 2$).

2. Theorem [Fu, 17.6]**.** *The map $F_{2g} \to \pi_1(\Sigma)$ given by $a_i \mapsto \alpha_i$ and $b_i \mapsto \beta_i$ for $i = 1, \ldots, g$ induces an isomorphism*

$$\pi_1(\Sigma) \cong F_{2g}/N_g.$$

Next consider cell decompositions of Σ, and in particular *triangulations*. Figure C.12 shows the standard 0, 1 and 2-simpleces σ_i, $i = 0$, 1 a d 2.
We also call any continuous map $\sigma_i \to \Sigma$ an i-simplex and denote it, by abuse of notation, as σ_i. A **triangulation** $\mathcal{T}$ consists of a family of i-simpleces glued along their edges, see Figures C.13 and C.14. Let V, E and F be the number of vertices, edges and faces respectively.

3. Theorem and Definition (Euler characteristic and triangulations) [Ki, Chapter 4.3]**,** [Ki, 5.9 and 5.15]**.** *We can always find a triangulation on Σ. The number*

$$\chi(\Sigma) = V - E + F$$

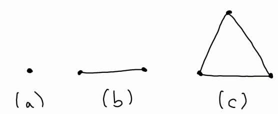

FIGURE 12. The standard (a) 0- (b) 1- (c) 2-simplex

is called the **Euler characteristic of** Σ *and does not depend on* $\mathcal{T}$*. Moreover, we have*

$$\chi(\Sigma) = 2g(\Sigma) - 2.$$

4. Examples.

(i) *The sphere* S^2*:* $\chi(S^2) = 2$, see Figure C.13

(ii) *The torus* $S^1 \times S^1$*:* $\chi(S^1 \times S^1) = 0$, see Figure C.14

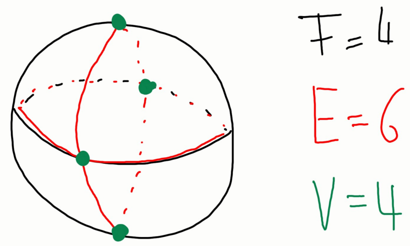

FIGURE 13. A triangulation of the sphere

Finally, we discuss the (co-)homology of surfaces. An i**-chain** c is a formal finite linear combination of i-simpleces, i.e.

$$c = \sum_{j=1}^{n} a_j \sigma_i^j$$

where $a_j \in \mathbb{Z}$. We can formally add chains and multiply them by integers. Let $C_i(\Sigma)$ the $\mathbb{Z}$-module of i-chains. In order to define a $\mathbb{Z}$-linear *differential* $\partial : C_i(\Sigma) \to C_{i+1}(\Sigma)$, $i = 0, 1$, we need to introduce *orientations*. We can orient a 1-simplex by choosing an initial and a final point. We can orient a 2-simplex by choosing clockwise or counter-clockwise orientation, see Figure C.15. The **boundary** $\partial\sigma_i$ of an oriented i-simplex is then the $i+1$-chain defined as follows:

$$\partial_i \sigma_i = \begin{cases} \text{terminal point } - \text{initial point}, & i = 1 \\ \epsilon(1)e_1 + \epsilon(2)e_2 + \epsilon(3)e_3 & i = 2, \end{cases}$$

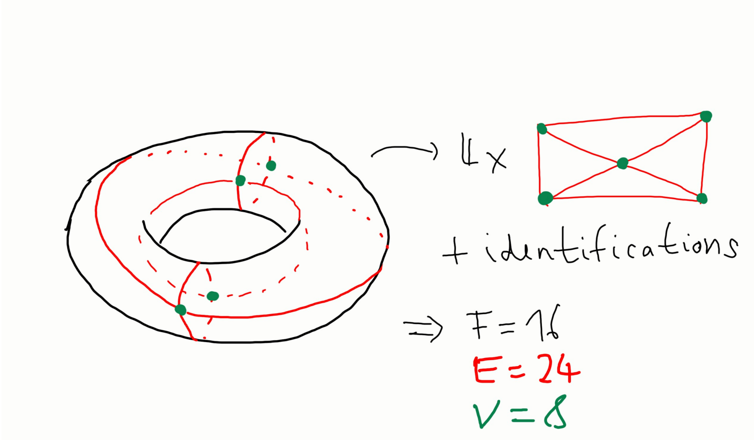

FIGURE 14. A triangulation of the torus

where $e_{1,2,3}$ are the (oriented) edges of σ_2 and where $\epsilon(i)$ is $+$ or $-$ according to whether or not the direction of the edge e_i is consistent with the direction of σ_2. We extend ∂ linearly to all of $C_i(\Sigma)$ so that ∂_i becomes a $\mathbb{Z}$-linear map $C_{i+1}(\Sigma) \to C_i(\Sigma)$.

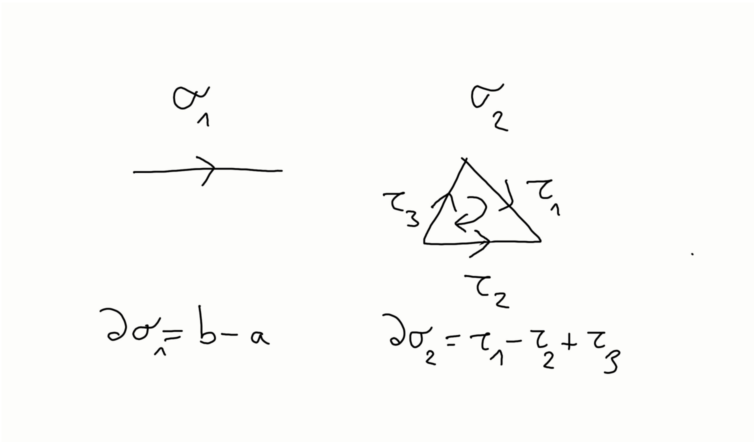

FIGURE 15. The boundary of the standard 1- and 2-simplex

5. Theorem [Ki, 6.10]. *$\partial_{i+1} \circ \partial_i = 0$, i.e. the boundary of a boundary chain is* 0. *We also say that $C_0 \xrightarrow{\partial} C_1 \xrightarrow{\partial} C_2$ defines a* **complex**.

For the next definition we extend the complex $\partial_i : C_i \to C_{i+1}$ by 0, i.e. we consider the complex

$$0 \to C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \to 0.$$

By the previous theorem, $\operatorname{im} \partial_{i+1} \subset \ker \partial_i$. We can therefore define:

6. Definition. The $\mathbb{Z}$-module

$$H_i(\Sigma, \mathbb{Z}) = \ker \partial_i / \operatorname{im} \partial_{i+1}$$

is called the ***i*-th homology group of** Σ.

This is again a topological invariant of Σ. It can be computed as follows.

7. Theorem [Ki, 6.20, 6.24 and 9.17]**.**

(i) $H_0(\Sigma, \mathbb{Z}) \cong \mathbb{Z}$, $H_2(\Sigma, \mathbb{Z}) \cong \mathbb{Z}$;

(ii) $H_1(\Sigma, \mathbb{Z}) \cong \pi_1(\Sigma)/[\pi_1(\Sigma), \pi_1(\Sigma)] \cong \mathbb{Z}^{2g(\Sigma)}$, *where* $[\pi_1(\Sigma), \pi_1(\Sigma)]$ *is the subgroup of* $\pi_1(\Sigma)$ *generated by elements of the form* $[a, b] = aba^{-1}b^{-1}$.

In particular, $\chi(\Sigma) = \dim_{\mathbb{Z}} H_0(\Sigma, \mathbb{Z}) - \dim_{\mathbb{Z}} H_1(\Sigma, \mathbb{Z}) + \dim_{\mathbb{Z}} H_2(\Sigma, \mathbb{Z})$.

8. Remark. Instead of $\mathbb{Z}$ we could take coefficients in any other (commutative) ring. In particular, if we take any other field k, then the previous theorem holds with $\mathbb{Z}$ being replaced by k (this is *false* if we consider homology on general topological spaces).

We can also consider the dual complex

$$0 \to C^0 := C_0^* \stackrel{d_0=\partial_1^*}{\to} C^1 := C_1^* \stackrel{d_1=\partial_1^*}{\to} C^2 := C_2^* \to 0.$$

Then $d_{i+1} \circ d_i = 0$ and we can define the **cohomology module**

$$H^i(\Sigma, \mathbb{Z}) = \ker d_i / \operatorname{im} d_{i-1}$$

(see also [MiSt, Appendix A] for a short introduction).

9. Theorem [MiSt, A.1]**.** *We have*

$$H^i(\Sigma, \mathbb{Z}) = \operatorname{Hom}_{\mathbb{Z}}(H_i(\Sigma, \mathbb{Z}), \mathbb{Z}).$$

In particular, $H^1(\Sigma, \mathbb{Z}) = \operatorname{Hom}_{\mathbb{Z}}(\pi_1(\Sigma), \mathbb{Z})$.

10. Remark.

(i) In general, only the defining complex of cohomology is dual to the complex of homology. The cohomology modules are not dual to the homology modules for general topological spaces.

(ii) As for homology we can replace $\mathbb{Z}$ by the fields $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$.

(iii) Homology and cohomology can be defined more generally for any topological space (though it might be a difficult invariant to compute), in particular noncompact surfaces. The only result we need in this course is that $H^2(\Sigma, \mathbb{Z}) = 0$ for any noncompact surface.

Appendix D. Field extensions

A **field extension** is an embedding $k \hookrightarrow K$ of the ground field k into some bigger field K (note in passing that any nontrivial k-linear map between fields is ncessarily injective). In particular, we may view K as a k vector space; it is customary to write K/k for the field extension and $[K : k]$ for $\dim_k K$, the **degree** of the field extension, but we will not do that. There are several types of field extensions which are important for us. A good reference is [Bo].

1. Definition (finite and algebraic field extensions). A field extension $k \subset K$ is **finite** if the dimension $\dim_k K < +\infty$. Moreover, $k \subset K$ is *algebraic* if for any $\alpha \in K$ there exists $f \in k[x]$ such that $f(\alpha) = 0$.

2. Proposition. *A finite field extension is algebraic.*

Proof. Indeed, if $\alpha \in K$, then there must be an n so that $\{1, \alpha, \alpha^2, \ldots, \alpha^n\}$ becomes linearly dependent over k, that is $\alpha^n = \sum_{i=0}^{n-1} a_i \alpha^i$. We let $k[\alpha]$ denote the subring of K generated by k and α, that is, $k[\alpha] = \{\sum_{i=0}^{n-1} a_i x^i \mid a_i \in k\}$. Since this is an integral domain and $k[x]$ Euclidean, so in particular a PID, the kernel of $k[x] \to k[\alpha]$, $X \mapsto \alpha$, must be a principal ideal, so $\ker = (f)$ for an irreducible element f. In particular, (f) is maximal so that $k[\alpha] = k(\alpha) := \operatorname{Quot} k[\alpha]$ is actually a field. Moreover, $\dim_k k(\alpha) = \deg f$. Indeed, $k[x]$ is Euclidean so that $g = qf + r$ with uniquely determined polynomials $\deg r < \deg f$. It follows that equivalence classes $1, \bar{x}, \bar{x}^2, \ldots, \bar{x}^{n-1}$ form a k-basis of $k[x]/(f) \cong k(\alpha)$. □

3. Remark. If in the proof of the previous proposition we normalise the polynomial f so that it is *monic*, i.e. $f = x^n + a_{n-1}x^{n-1} + \ldots + a_0$, then f is called the **minimal polynomial** of α and is uniquely determined. In general, if $f \in k[x]$ is irreducible, then $k \subset k[x]/(f)$ is a finite extension in which f has a root.

4. Examples.

(i) Let $k = \mathbb{R}$ and $f = x^2 + 1$, then $\mathbb{C} = \mathbb{R}[x]/(x^2 + 1)$.

(ii) $\bar{\mathbb{Q}} = \{\alpha \in \mathbb{C} \mid \alpha \text{ algebraic over } \mathbb{C}\}$ be the *algebraic closure of* $\mathbb{Q}$. Then $\mathbb{Q}(\sqrt[n]{3}) \subset \bar{\mathbb{Q}}$ has minimal polynomial $X^n - 3$ since it is irreducible by Eisenstein's criterion. It follows that $\dim_{\mathbb{Q}} \mathbb{Q}(\sqrt[n]{3}) = n$. In particular, $\dim_{\mathbb{Q}} \bar{\mathbb{Q}} = \infty$ which shows that algebraic extensions need not be finite in general.

As the first example shows, a field k need not be algebraically closed, i.e. there are polynomials $f \in k[x]$ which do not admit a root in k. However, we have the following

5. Theorem (existence of the algebraic closure). *For any field k there exists an algebraic field extension $k \subset K$ such that K is algebraically closed field.*

Proof. See [Bo, Theorem 3.4.4]. □

Item (ii) in the previous example can be generalised as follows:

6. Definition. If k is a field and K an algebraically closed field so that $k \subset K$ is algebraic, we call

$$\bar{k} = \{\alpha \in K \mid \alpha \text{ is algebraic over } k\}$$

the **algebraic closure of** k. The field $\bar{k}$ is determined up to isomorphism which restricts to the identity on k (cf. [Bo, Corollaries 3.4.7 and 10]).

7. Definition (Galois extensions). A field extension $k \subset K$ is **normal** if any irreducible polynomial $f \in k[x]$ which has a root in K splits into linear factors in $K[x]$. Further, $k \subset K$ is called **separable** if it is algebraic and every $a \in K$ is the root of a separable polynomial in $k[x]$, i.e. a polynomial whose roots are simple. A field extension is **Galois** if it is normal and separable. In this case, the group of automorphisms of K which leave k fixed is called the **Galois group**of the field extension $k \subset K$ and written $\operatorname{Gal}(K/k)$.

The central theorem of Galois theory is this:

8. Theorem (Galois). *Let $k \subset K$ be a finite Galois extension $\Rightarrow$ there is a correspondence between subgroups of* $\operatorname{Gal}(K/k)$ $\leftrightarrow$ *fields $k \subset L \subset K$. More* *precisely, given* $H \subset \mathrm{Gal}(K/k)$*,* L *is the field fixed by the endomorphism* $\sigma : K \to K$ *in* $H \subset \mathrm{Gal}(K/k)$ *while given a field* $k \subset L \subset K$ *we get the subgroup* $\mathrm{Gal}(K/L)$*.*

Proof. See [Bo, Theorem 4.1.6]. □

Next we consider separable field extensions.

9. Definition. A field k is called **perfect** if any algebraic field extension of k is separable.

In characteristic 0 every algebraic field extension is separable [Bo, Remark 3.6.4], since any irreducible polynomial over a field of characteristic 0 is separable [Bo, Proposition 3.6.2]. Hence any such field is perfect. Further examples are finite fields or algebraically closed fields are also perfect. One of the main features of finite separable extensions is the

10. Theorem of the Primitive element. *If* $k \subset K$ *is a finite separable field extension, then there exists a so-called* **primitive element** $\alpha \in K$ *such that* $K = k(\alpha)$*.*

Proof. See [Bo, Proposition 3.6.12] □

Next we consider non-algebraic field extensions.

11. Definition (transcendence base). Consider a field extension $k \subset K$. Elements $\alpha_1, \ldots \alpha_n \in K$ are **algebraically independent** if the natural surjection

$$k[x_1, \ldots, x_n] \to k[\alpha_1, \ldots, \alpha_n] \subset K \to 0$$

sending x_i to α_i is actually an isomorphism of k-algebras, that is, we have an injection $k[x_1, \ldots, x_n] \hookrightarrow K$ sending x_i to α_i. Put differently, if there is a polynomial relation of the form $f(\alpha_1, \ldots, \alpha_n) = 0$ for $f \in k[x_1, \ldots, x_n]$, then $f = 0$. A family $\mathfrak{B} = \{\alpha_i\}_{i \in I}$ is algebraically independent if the previous definition applies for any finite subset of $\mathfrak{B}$. If in this case the field extension $k(\mathfrak{B}) \subset K$ is algebraic, then A is called a **transcendence base**. If $K = k(\mathfrak{B})$ for some transcendence base, we call the field extension $k \subset K$ **purely transcendental**. A finite field extension of a purely transcendental one defines a so-called **function field**.

Any field extension $k \subset K$ can be factorised into a purely transcendental field extension $k \subset k(\mathfrak{B}) \subset K$, where the latter field extension is algebraic:

12. Proposition and Definition (transcendence degree). *Any field extension* $k \subset K$ *admits a transcendence base. Any two transcendence bases have the same cardinality which we call the* **transcendence degree** *and write* $\mathrm{trdeg}_k K$*.*

Proof. See [Bo, Proposition 7.1.3 and Theorem 7.1.5]. □

13. Proposition (Zariski's lemma). *Let* $k \subset K$ *be a field extension, where* K *is a finitely generated* k*-algebra. Then* $k \subset K$ *is a finite field extension.*

Proof. Let $K = k[\alpha_1, \ldots, \alpha_n]$. If K is algebraic over k, we are done. So assume otherwise and relabel the α_i in such a way that $x_1, \ldots, x_r$ are algebraically independent over k, and x_i are algebraic over the field $L = k(\alpha_1, \ldots, \alpha_r)$. Hence K is a finite algebraic extension of L and therefore a finite L-module. By Noether normalisation, L is a finitely generated k-algebra, that is, $L = k[\beta_1, \ldots, \beta_s]$. But this can only happen if $L = k$. To see this rigourosly, we note that each $\beta_i \in L$ so that $\beta_i = f_i/g_i$ for polynomials f_i and g_i in $x_1, \ldots, x_r$. Now there are infinitely many irreducibles in the factoriel ring $k[x_1, \ldots, x_r]$ (there are infinitely many primes just by the same argument as for $\mathbb{Z}$). Hence there is an irreducible polynomials which is prime to any of the finitely many g_i (for instance, take $h = g_1 \cdot \ldots \cdot g_s + 1$ would do). Therefore, $h^{-1} \in L$ cannot be a polynomial in the y_i (clear the common denominator and multiply by h). Contradiction. □

Do not confuse the notion of a finitely generated k-algebra K with a finitely generated field extension $k \subset K$. If K is a finitely generated k-algebra, then there exist $\alpha_i \in K$ such that $K = k[\alpha_1, \ldots, \alpha_n]$. The previous proposition then says that no subset of these generators is algebraically independent. If $k \subset K$ is a finitely generated field extension, then $K = k(\alpha_1, \ldots, \alpha_r)$ where we can label the α_i in such a way that $\alpha_1, \ldots, \alpha_n$ form a transcendence base so that $k(\alpha_1, \ldots, \alpha_n) \subset K$ is an algebraic, in fact finite extension of the *purely transcendental field extension* $k \subset k(\alpha_1, \ldots, \alpha_n)$.

14. Proposition and definition (separably generated field extensions). *A field extension $k \subset K$ is* **separably generated** *if there is a transcendence base $\mathfrak{B}$ such that $k(\mathfrak{B}) \subset K$ is a separable algebraic extension. In this case, $\mathfrak{B}$ is called a* **separating transcendence base**. *For a finitely and separably generated field extension $k \subset K = k(\alpha_1, \ldots, \alpha_r)$ the set of generators $\{\alpha_i\}$ contains a separating transcendence base.*

Proof. See [Bo, Proposition 7.3.7] □

15. Proposition (perfect fields and separably generated field extensions). *If k is a perfect field, any finitely generated field extension $k \subset K$ is separably generated.*

Proof. See [Bo, Corollary 3.7.8]. □

Appendix E. Category theory

We discuss the basic notions of category theory. For a further development see for instance [GeMa].

1. Definition (category). A **category** $\mathcal{C}$ consists of the following data:

(i) A class of **objects** $\mathrm{Ob}\,\mathcal{C}$;
(ii) for any two objects A, $B \in \mathrm{Ob}\,\mathcal{C}$ a *set* $\mathrm{Mor}_{\mathcal{C}}(A, B)$ of *morphisms*. We denote an element of $\mathrm{Mor}_{\mathcal{C}}(A, B)$ usually by $A \to B$.

Furthermore, for any three objects A, B and $C \in \mathcal{C}$ there exists a map

$$\circ : \mathrm{Mor}_{\mathcal{C}}(A, B) \times \mathrm{Mor}_{\mathcal{C}}(B, C) \to \mathrm{Mor}_{\mathcal{C}}(A, C), \quad (f, g) \mapsto g \circ f$$

such that $\mathrm{Mor}_{\mathcal{C}}(A, B)$ is a monoid, i.e.

(i) $\circ$ is *associative*, i.e. $(g \circ f) \circ h = g \circ (f \circ h)$;
(ii) for all $A \in \operatorname{Ob}\mathcal{C}$ there exists a morphism $\mathrm{Id}_A \in \operatorname{Mor}_{\mathcal{C}}(A, A)$, the so-called **identity** of A such that for all $B \in \operatorname{Ob}\mathcal{C}$ and for all $f \in \operatorname{Mor}_{\mathcal{C}}(A; B)$ and $g \in \operatorname{Mor}_{\mathcal{C}}(B, A)$ we have
$$f \circ \mathrm{Id}_A = f \quad \text{ and } \mathrm{Id}_A \circ g = g.$$

To simplify the notation we often write Mor instead of $\operatorname{Mor}_{\mathcal{C}}$. A category $\mathcal{C}$ is **small** if $\operatorname{Ob}\mathcal{C}$ is a set.

2. Definition (isomorphism). Let $\mathcal{C}$ be a category. A morphism $f \in \operatorname{Mor}_{\mathcal{C}}(A, B)$ is called a **(categorical) isomorphism** if there exists $g \in \operatorname{Mor}_{\mathcal{C}}(B, A)$ such that $g \circ f = \mathrm{Id}_A$ and $f \circ g = \mathrm{Id}_B$, that is, f has a two sided inverse. In this case we also write $g = f^{-1}$. If $\mathcal{C}$ is small, then being isomorphic defines an equivalence relation on $\operatorname{Ob}\mathcal{C}$ and we denote by $\operatorname{Iso}(\mathcal{C})$ the set of equivalence classes.

3. Examples. (see also [GeMa, Section II.§1.5] for examples.)

(i) The basic example is the category **SET** of sets with maps as morphisms. Note that there is no set of sets (cf. Russell's paradoxon) which is why the objects form a class, not a set. On the other hand, $\operatorname{Mor}_{\mathbf{S}ET}(A, B) \subset A \times B$ is of course a set. Isomorphisms are just bijective maps. Further examples in this vein are given by algebraic categories such as the category of abelian groups **ABG** or A-modules $\mathbf{MOD}_A$ with the corresponding notion of (iso)morphisms (group morphisms, A-linear (bijective) maps, etc.) or geometric categories (e.g. category of varieties with (bi)regular maps as (iso)morphisms). This also explains the general notation $A \to B$ for morphisms.
(ii) More exotic examples include the catgeory $\mathcal{C}(I)$ of a partially ordered set I, where $\operatorname{Ob}\mathcal{C}(I) = I$, and $\operatorname{Mor}_{\mathcal{C}(I)}(i, j)$ consists of one element if $i \leqslant j$ and is empty otherwise. In particular, $\operatorname{Mor}_{\mathcal{C}(I)}(i, i) = \{\mathrm{Id}_i\}$ and an element $f \in \operatorname{Mor}_{\mathcal{C}}(i, j)$ is an isomorphism if and only if $i = j$ and $f = \mathrm{Id}_i$. If X is a topological space we can consider the category $\mathbf{TOP}_X$. Here, the objects are the open subsets of X (a subset of the power set of X), and $\operatorname{Mor}(U, V)$ is the inclusion if $U \subset V$ and the empty set otherwise. Again, $\operatorname{Mor}(U, U) = \mathrm{Id}_U$ and $f \in \operatorname{Mor}(U, V)$ is an isomorphism if and only if $U = V$ and $f = \mathrm{Id}_U$. Finally, we can consider the category $\mathbf{SHEAF}_X$ whose objetcs are sheaves on X, and $\operatorname{Mor}(\mathcal{F}, \mathcal{G})$ are sheaf morphisms. Here, the notion of isomorphism is the catgeorical one, i.e. $\varphi : \mathcal{F} \to \mathcal{G}$ is an isomorphism of sheaves if and only if it has a two sided inverse (cf. Definition ??.??). The definition of injective and surjective sheaf morphism was designed in such a way that an isomorphism is precisely a morphism which is injective and surjective, cf. Exercise ??.??.

4. Definition. An object U of a category is called **universally repelling** (**attractive**) if for any other object A there exists exactly one morphism $U \to A$ ($A \to U$). For sake of brievety we also call U simply **universal**.

5. Remark. It follows immediately from the definition that if U is universal, then $\operatorname{Mor}(U, U) = \{\mathrm{Id}_U\}$, and U is unique up to *unique* isomorphism.

6. Example.

(i) Let $M_1, \ldots, M_r$ be a finite number of A-modules. We construct a categaory $\mathcal{C}$ as follows. Take r-multilinear maps from $f : M_1 \times \ldots \times M_r \to N$, where N is some further A-module, as the objects of our category $\mathcal{C}$. For two objects $f : M_1 \times \ldots \times M_r \to N$, $g : M_1 \times \ldots \times M_r \to L$, let a morphism $f \to g \in$

Mor(f, g) be an A-linear map $h : N \to L$ such that $g = l \circ f$. Then the tensor product is a universally repelling object for $\mathcal{C}$.

(ii) Let X be a topological space which admits a universal covering space $p : \tilde{X} \to X$. This is a universally repelling object in the category of covering maps of X whose objects are covering maps $p : Y \to X$, and whose morphisms between two covering maps $p : Y \to X$ and $q : Z \to X$ are continuous maps $f : Y \to Z$ such that $q \circ f = p$.

We can also consider "maps" between categories.

7. Definition (functor). For two categories $\mathcal{C}$ and $\mathcal{D}$ we call $F : \mathcal{C} \to \mathcal{D}$ a **functor** an assignement which associates with any object A in $\mathcal{C}$ an object $F(A)$ in $\mathcal{D}$, and for any two objects A and B a map $\mathrm{Mor}_{\mathcal{C}}(A, B) \to \mathrm{Mor}_{\mathcal{D}}(F(A), F(B))$ (F is **covariant**) or $\mathrm{Mor}_{\mathcal{C}}(A, B) \to \mathrm{Mor}_{\mathcal{D}}(F(B), F(A))$ (F is **contravariant**) taking f to $F(f)$, and having the following properties:

(i) $F(\mathrm{Id}_A) = \mathrm{Id}_{F(A)}$;

(ii) $F(f \circ g) = F(f) \circ F(g)$ (F covariant) or $F(f \circ g) = F(g) \circ F(f)$ (F contravariant);

(iii) A presheaf onb X can be regarded as a contravariant functor $\mathbf{Top}_X \to \mathbf{AbG}$.

8. Remark. If F is a covariant (contravariant) functor, we often write f_* (f^*) for $F(f)$.

9. Examples.

(i) The basic example of a covariant functor is the so-called **forgetful functor** from a category $\mathcal{C}$ to **Set** which associates with say an A-module its underlying set,and with an A-linear map its underlying set theoretic map.

(ii) The assignement which takes an A-module M to its dual module $M^\vee$, and an A-linear map $f : M \to N$ to the dual map $f^\vee : N^\vee \to M^\vee$ defined by $f(\lambda)(m) = \lambda(f(m))$ for all $m \in M$ is a contravariant functor.

(iii) Consider the category $\mathbf{TOP}_*$ of pointed topological spaces (X, a) as objects together with continuous maps between them as morphisms, i.e. $f : (X, a) \to (Y, b)$ satisfies $f(a) = b$. The assignement $(X, a) \mapsto \pi_1(X, a) =$ the fundamental group of X, $f : (X, a) \to (Y, b) \mapsto f_* : \pi_1(X, a) \to \pi_1(Y, b)$ is a functor between $\mathbf{TOP}_*$ and **GRP**, the category of groups.

A useful notion of "isomorphic" catgeories is this.

10. Definition (equivalence of categories). Two small categories $\mathcal{C}$ and $\mathcal{D}$ are **(covariantly) equivalent** if there exists a covariant functor $F : \mathcal{C} \to \mathcal{D}$ such that F

(i) induces a surjective map on isomorphism classes $\mathrm{Iso}(\mathcal{C}) \to \mathrm{Iso}(\mathcal{D})$. Put differently, for any object y in $\mathcal{D}$ there exists an object x in $\mathcal{C}$ with $F(x)$ is isomorphic with y.

(ii) *full and faithful*, that is, for any two objects x_1, x_2 in $\mathcal{C}$ the induced map $F(x_1, x_2) : \mathrm{Mor}(x_1, x_2) \to \mathrm{Mor}(F(x_1), F(x_2))$ is surjective and injective.

An analogous definition applies for **contravariant equivalent** categories.

11. Example. The category of affine varieties over k is equivalent with the category of finitely generated k-algebras without zero divisors.

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